# CHARACTERIZATION OF ALMOST $L^p$-EIGENFUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR

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Abstract. In [36] Roe proved that if a doubly-infinite sequence $\{f_k\}$ of functions on $\mathbb{R}$ satisfies $f_{k+1} = (df_k/dx)$ and $|f_k(x)| \leq M$ for all $k = 0, \pm 1, \pm 2, \cdots$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x + \varphi)$ where $a$ and $\varphi$ are real constants. This result was extended to $\mathbb{R}^n$ by Strichartz [41] where $d/dx$ is substituted by the Laplacian on $\mathbb{R}^n$. While it is plausible to extend this theorem for other Riemannian manifolds or Lie groups, Strichartz showed that the result holds true for Heisenberg groups, but fails for hyperbolic 3-space. This negative result can be indeed extended to any Riemannian symmetric space of noncompact type. We observe that this failure is rooted in the $p$-dependance of the $L^p$-spectrum of the Laplacian on the hyperbolic spaces. Taking this into account we shall prove that for all rank one Riemannian symmetric spaces of noncompact type, or more generally for the harmonic $NA$ groups, the theorem actually holds true when uniform boundedness is replaced by uniform "almost $L^p$ boundedness". In addition we shall see that for the symmetric spaces this theorem is capable of characterizing the Poisson transforms of $L^p$ functions on the boundary, which some what resembles the original theorem of Roe on $\mathbb{R}$.

Contents



## 1. Introduction

This paper revolves mainly around results characterizing eigenfunctions of the Laplace-Beltrami operator $\Delta$ on Riemannian symmetric spaces of noncompact type with real rank one (which we shall denote by $X$) and its nonsymmetric generalizations namely the Damek-Ricci (DR) spaces (which will be denoted by $S$) in which the former spaces account for a very thin sub class (see [3]). DR spaces are also known as harmonic $NA$ groups. They are solvable Lie groups as well as harmonic manifolds and appear as counter examples to the Lichnerowicz conjecture in the noncompact case (see [9]).

We are concerned about the following generalization of Roe's theorem proved by Strichartz ([41]) which involves the Laplace operator $\Delta_{\mathbb{R}^n}$ of $\mathbb{R}^n$. (See also [19, 20, 26] and the references therein.)

**Theorem 1.0.1** (Strichartz). *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of functions on $\mathbb{R}^n$ with $\|f_k\|_{L^\infty(\mathbb{R}^n)} \leq M$ for all $k \in \mathbb{Z}$. If for some $\alpha > 0$, $\Delta_{\mathbb{R}^n} f_k = \alpha f_{k+1}$ for all $k \in \mathbb{Z}$, then $\Delta_{\mathbb{R}^n} f_0 = -\alpha f_0$.*

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The case $\alpha = 1$ was proved in [41], but the same proof works for other values of $\alpha$ as well. It is not difficult to observe that the theorem above holds true if one replaces $L^\infty(\mathbb{R}^n)$ by $L^p(\mathbb{R}^n)$-norm with the restriction $p > 2n/(n-1)$ for the result to be non-vacuous (see [1]) or by weak $L^p$-norm for $p \geq 2n/(n-1)$ (which can be substantiated by standard estimate of Bessel functions (see [38])). Our starting point however is a striking counter example in [41] which shows that the result above is no longer true if $\mathbb{R}^n$ is replaced by the symmetric space $SL(2,\mathbb{C})/SU(2)$. Precisely, there exists a uniformly bounded doubly infinite sequence $\{f_k\}_{k\in\mathbb{Z}}$ of radial eigenfunctions of $\Delta$ on $SL(2,\mathbb{C})/SU(2)$ satisfying $\Delta f_k = f_{k+1}$ but $\Delta f_0 \neq -f_0$. This counter example can be strengthened. Precisely, in any $X$ or $S$ a sequence $\{f_k\}$ can be constructed which satisfies the hypothesis of the theorem above with uniformly bounded (or uniformly bounded with respect to $L^p$-norm, $2 < p < \infty$), but $f_0$ is not even an eigenfunction of $\Delta$ (see section 3). This motivated us to have a detailed investigation of the phenomenon in the context of symmetric or DR spaces. A somewhat deeper understanding of the counter example mentioned above tells us that the failure of the result for hyperbolic spaces can be ascribed to the, by now well known, fact that the $L^p$-spectrum of the Laplacian on $X$ or on $S$ depends on $p$ (see [32, 42, 3]). This is one of the most intriguing difference between analysis on Lie groups with polynomial growth and that of with exponential growth (see, for instance [21, 22]).

Among other things in the present paper we shall obtain an essentially sharp version of Theorem 1.0.1 on symmetric and on DR spaces which will involve various uniform size-estimates close to $L^p$, instead of uniform boundedness. These size-estimates arise naturally due to the behavior of Poisson transforms (of $L^p$-functions on the boundary) whose eigenvalues lie on the boundary of the $L^p$-spectrum of $\Delta$ (see [33, 30, 35] and section 3.1) and among them weak $L^p$-norm can be singled out by its translation invariance. We mention here two representative theorems (for notation see section 2). We shall prove these results as a consequence of a general version of Theorem 1.0.1 on DR spaces $S$ in section 5.

**Theorem A.** *Let $f$ be a measurable functions on $S$ and $\alpha$ a nonzero real number. If $\|\Delta^k f\|_{2,\infty} \leq M(\alpha^2+\rho^2)^k$ for all $k \in \mathbb{Z}$, for some $M > 0$, then $\Delta f = -(\alpha^2+\rho^2)f$. In particular when $S = X$ is an Iwasawa $NA$ group, then $f = \mathcal{P}_\alpha F$ for some $F \in L^2(K/M)$.*

**Theorem B.** *Let $f$ be a measurable functions on $S$ and $p \in (1,2)$. If $\|\Delta^k f\|_{p',\infty} \leq M(4\rho^2/pp')^k = M((i\gamma_{p'}\rho)^2+\rho^2)^k$ for $k = 0,1,2,\ldots$, for some $M > 0$, then $\Delta f = -(4\rho^2/pp')f$. In particular when $S = X$ is an Iwasawa $NA$ group, then $f = \mathcal{P}_{-i\gamma_p\rho}F$ for some $F \in L^{p'}(K/M)$.*

If we define $f_k = (4\rho/pp')^{-k}\Delta^k f$ then the statements of these theorems resemble Theorem 1.0.1. We observe en passant that Theorem A and B have structural resemblance with the following celebrated result of Kotake and Narasimhan [29]:

**Theorem 1.0.2.** *Let $\Omega$ be an open set of $\mathbb{R}^n$. Let $A$ be a linear elliptic operator of order $m$ with analytic coefficients in $\Omega$. If a $C^\infty$ function $f$ satisfies for some $c > 0$, $\|A^k f\|_{L^2(\Omega)} \leq (mk)!c^{k+1}$ for all nonnegative integers $k$, then $f$ is real analytic on $\Omega$.*

It is not difficult to see that using the $G$-invariance of $\Delta$ on $X = G/K$, it is enough to restrict to the $K$-isotypic components of $f$. However the lack of *rotation* on the DR spaces makes it more interesting and forces us to adopt a different approach.

We may stress that in the theorems above, for the symmetric spaces, a concrete description of the eigenfunction $f$ as Poisson transforms (of $L^p$-functions on the boundary) is achieved. A crucial ingredient for this, in the case of Theorem B is a characterization of eigenfunctions due to Lohoué and Rychener ([33], see also [37]). The corresponding result required for Theorem A seems to be new

and will be proved in section 4 using a result of Ionescu ([23]). A rich body of literature concerning representation theorems of eigenfunctions of $\Delta$ on $X$ (see e.g. [12, 28, 39, 37, 6, 5], see also [8]) is already available. In section 4 we shall briefly survey the existing results in this direction and further generalize them, keeping our need in view. In particular we shall settle a question posed in [6]. This section may be of independent interest and is independent of the rest of the paper. Some of these results will be used to obtain analogues of Theorems A and Theorem B.

The phrase *almost* $L^p$ is used in this paper to mean the size estimates which are close to $L^p$-norm, weak $L^p$ being one of the examples. In section 4 we shall obtain others. We shall use these estimates to formulate various analogues of Theorem A and Theorem B. For motivation we cite a version of Theorem 1.0.1 on $\mathbb{R}^n$ which involves the following *almost* $L^2$-size estimate:

$$M_2(f) = \left(\limsup_{R\to\infty} \frac{1}{R}\int_{B(0,R)} |f(x)|^2 dx\right)^{1/2},$$

where $B(0,R)$ is the ball of radius $R$ in $\mathbb{R}^n$, centered at origin.

**Theorem 1.0.3.** *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of measurable functions on $\mathbb{R}^n$ satisfying $\Delta f_k = \alpha^2 f_{k+1}$ for some $\alpha \in \mathbb{R}$ and $M_2(f_k) \leq M$ for all $k \in \mathbb{Z}$. Then there exists $F \in L^2(S^{n-1})$ such that $f_0(x) = \int_{S^{n-1}} e^{i\alpha\langle x,\omega\rangle} F(\omega) d\omega$.*

The result above is an easy consequence of Lemma 3.2 in [40] and the idea of the proof of Theorem 1.0.1. The details is left to the interested readers. As mentioned earlier, unlike uniform boundedness, the size estimate used in the formulation above is not translation invariant. An analogue of this result for symmetric spaces and for more general $p$ will be proved in section 6. We note here that on DR spaces, concrete realization of a function (satisfying the hypothesis of Theorem A or Theorem B) as Poisson transform of a function on its boundary seems to be more involved and is still open.

There are important eigenfunctions of $\Delta$, e.g. the (powers of) Poisson kernel: $e^{(i\alpha+\rho)H(x^{-1}k)}$, which are the objects analogous to the functions $x \mapsto e^{i\langle\lambda,x\rangle}$ on $\mathbb{R}^n$. However they do not belong to any $L^p$, weak-$L^p$ or in general in any Lorentz spaces. They are not even bounded, unlike their Euclidean counter parts. In particular they do not satisfy the hypotheses of Theorem A or Theorem B. One of the purposes of the formulations of the Theorems in section 5 is to prevent their a priori exclusions, where we shall weaken the hypothesis suitably using $L^p$-tempered distributions.

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## 2. Preliminaries

The preliminaries and notation related to the semisimple Lie groups and the associated symmetric spaces are standard and can be found for example in [17], while that related to DR spaces can be retrieved from [4, 3, 35, 30]. To make the article self-contained we shall gather only those results which are required for this paper.

2.1. **Generalities.** For any $p \in [1,\infty)$, let $p' = p/(p-1)$. The letters $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ denote respectively the set of integers, real numbers, complex numbers and quaternions. For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote respectively the real and imaginary parts of $z$. We denote the nonzero real numbers and nonnegative integers respectively by $\mathbb{R}^\times$ and $\mathbb{Z}^+$. For a set $A$ in a measure space we shall use $|A|$ to denote the measure of $A$. We shall follow the standard practice of using the letters $C, C_1, C_2$ etc. for positive constants, whose value may change from one line to another. Occasionally the constants will be suffixed

to show their dependencies on important parameters. Everywhere in this article the symbol $f_1 \asymp f_2$ for two positive expressions $f_1$ and $f_2$ means that there are positive constants $C_1, C_2$ such that $C_1 f_1 \leq f_2 \leq C_2 f_1$.

Apart from the Lebesgue spaces we also need to deal with the Lorentz spaces which we shall introduce briefly (see [13, 38, 35] for details). Let $(M, m)$ be a $\sigma$-finite measure space, $f : M \longrightarrow \mathbb{C}$ be a measurable function and $p \in [1, \infty)$, $q \in [1, \infty]$. We define

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [f^*(t)t^{1/p}]^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty, \\ \\ \sup_{t>0} t d_f(t)^{1/p} = \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty, \end{cases}$$

where for $\alpha > 0$, $d_f(\alpha) = |\{x \mid f(x) > \alpha\}|$ is the distribution function of $f$ and $f^*(t) = \inf\{s \mid d_f(s) \leq t\}$ is the *decreasing rearrangement* of $f$. We take $L^{p,q}(M)$ to be the set of all measurable $f : M \longrightarrow \mathbb{C}$ such that $\|f\|^*_{p,q} < \infty$. For $1 \leq p < \infty$, $L^{p,p}(M) = L^p(M)$ and $\|\cdot\|^*_{p,p} = \|\cdot\|_p$. By $L^{\infty,\infty}(M)$ and $\|\cdot\|^*_{\infty,\infty}$ we mean respectively the space $L^\infty(M)$ and the norm $\|\cdot\|_\infty$. The space $L^{p,\infty}(M)$ is known as the weak $L^p$-space. Following properties of the Lorentz spaces will be required:

(i) For $1 < p, q < \infty$, the dual space of $L^{p,q}(M)$ is $L^{p',q'}(M)$ and the dual of $L^{p,1}(M)$ is $L^{p',\infty}(M)$.
(ii) If $q_1 \leq q_2 \leq \infty$ then $L^{p,q_1}(M) \subset L^{p,q_2}(S)$ and $\|f\|^*_{p,q_2} \leq \|f\|^*_{p,q_1}$.

The Lorentz "norm" $\|\cdot\|^*_{p,q}$ is indeed only a quasi-norm and this makes the space $L^{p,q}(M)$ a quasi Banach space (see [13, p. 50]). However for $1 < p \leq \infty$, there is an equivalent norm $\|\cdot\|_{p,q}$ which makes it a Banach space (see [38, Theorems 3.21, 3.22]). We shall slur over this difference and use the notation $\|\cdot\|_{p,q}$.

2.2. **Damek-Ricci spaces.** Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a $H$-type Lie algebra where $\mathfrak{v}$ and $\mathfrak{z}$ are vector spaces over $\mathbb{R}$ of dimensions $m$ and $l$ respectively. Indeed $\mathfrak{z}$ is the centre of $\mathfrak{n}$ and $\mathfrak{v}$ is its ortho-complement with respect to the inner product of $\mathfrak{n}$. Then we know that $m$ is even. The group law of $N = \exp \mathfrak{n}$ is given by

$$(X, Y).(X', Y') = ((X + X', Y + Y' + \frac{1}{2}[X, X']) \quad X \in \mathfrak{v}, Y \in \mathfrak{z}.$$

We shall identify $\mathfrak{v}$ and $\mathfrak{z}$ and $N$ with $\mathbb{R}^m$, $\mathbb{R}^l$ and $\mathbb{R}^m \times \mathbb{R}^l$ respectively. The group $A = \{a_t = e^t \mid t \in \mathbb{R}\}$ acts on $N$ by nonisotropic dilation: $\delta_t(X, Y) = (e^{t/2}X, e^t Y)$. Let $S = NA = \{(X, Y, a_t) \mid (X, Y) \in N, t \in \mathbb{R}\}$ be the semidirect product of $N$ and $A$ under the action above. The group law of $S$ is thus given by:

$$(X, Y, a_t)(X', Y', a_s) = (X + a_{t/2}X', Y + a_t Y' + \frac{a_{t/2}}{2}[X, X'], a_{t+s}).$$

It then follows that $\delta_t(X, Y) = a_t n a_{-t}$, where $n = (X, Y)$. The Lie group $S$ is solvable, connected and simply connected with Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$. It is well known that $S$ is nonunimodular. The homogenous dimension of $S$ is $Q = m/2 + l$. For convenience we shall also use the notation $\rho = Q/2$. The group $S$ is equipped with the left-invariant Riemannian metric $d$ induced by

$$\langle (X, Z, \ell), (X', Z', \ell') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + \ell\ell'$$

on $\mathfrak{s}$. The associated left invariant Haar measure $dx$ on $S$ is given by

$$\int_S f(x)dx = \int_{N \times A} f(na_t)e^{-Qt} dt dn, \tag{2.2.1}$$

where $dn(X, Y) = dX\, dY$ and $dX, dY, dt$ are Lebesgue measures on $\mathfrak{v}$, $\mathfrak{z}$ and $\mathbb{R}$ respectively. We denote the Laplace-Beltrami operator associated to this Riemannian structure by $\Delta$.

The group $S$ can also be realized as the unit ball

$$B(\mathfrak{s}) = \{(X, Z, \ell) \in \mathfrak{s} \mid |X|^2 + |Z|^2 + \ell^2 < 1\}$$

via a Cayley transform $C : S \longrightarrow B(\mathfrak{s})$ (see [3, p. 646–647] for details). For an element $x \in S$, let

$$|x| = d(C(x), 0) = d(x, e) = \log \frac{1 + \|C(x)\|}{1 - \|C(x)\|}.$$

In particular $d(a_t, e) = |t|$. The left Haar measure in geodesic polar coordinates is given by ([3, (1.16)])

$$dx = 2^m (\sinh r)^k (\sinh \frac{r}{2})^m \, dr \, d\omega \tag{2.2.2}$$

where $r = |x|$ and $d\omega$ denotes the surface measure on the unit sphere $\partial B(\mathfrak{s})$ in $\mathfrak{s}$. For convenience we shall write the corresponding integral formula as $\int_S f(x)dx = \int_0^\infty \int_{\partial B(\mathfrak{s})} f(rw)J(r)dr \, d\omega$. A function $f$ on $S$ is called *radial* if for all $x, y \in S$, $f(x) = f(y)$ if $d(x, e) = d(y, e)$. By abuse of notation we shall sometimes consider a radial function $f$ as a function of $|x|$ and for such a function $\int_S f(x)dx = \int_0^\infty f(r)J(r)dr$. For a function space $\mathcal{L}(S)$ on $S$ we denote its subspace of radial functions by $\mathcal{L}(S)^\#$. For a suitable function $f$ on $S$ its radialization $Rf$ is defined as

$$Rf(x) = \int_{S_\nu} f(y) d\sigma_\nu(y), \tag{2.2.3}$$

where $\nu = |x|$ and $d\sigma_\nu$ is the surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S \mid d(y, e) = \nu\}$ normalized by $\int_{S_\nu} d\sigma_\nu(y) = 1$. It is clear that $Rf$ is a radial function and if $f$ is radial then $Rf = f$. The following properties of the radialization operator will be needed (see [10, 4]):

(1) $\langle R\phi, \psi\rangle = \langle \phi, R\psi\rangle \quad \phi, \psi \in C_c^\infty(S)$,
(2) $R(\Delta f) = \Delta(Rf)$.

Since by (1) above, $\int_S f(x)dx = \int_0^\infty Rf(r)J(r)dr$, we have $\|Rf\|_1 \leq \|f\|_1$. Interpolating ([38, p. 197]) with the trivial $L^\infty$-boundedness of $R$ we have,

$$\|Rf\|_{q,r} \leq \|f\|_{q,r}, 1 < q < \infty, 1 \leq r \leq \infty.$$

To proceed towards the Fourier transform we need to introduce the notion of Poisson kernel. The Poisson kernel $\wp : S \times N \longrightarrow \mathbb{R}$ is given by $\wp(x, n) = \wp(n_1 a_t, n) = P_{a_t}(n^{-1}n_1)$ where

$$P_{a_t}(n) = P_{a_t}(X, Y) = C a_t^Q \left( \left( a_t + \frac{|X|^2}{4} \right)^2 + |Y|^2 \right)^{-Q}, \ n = (X, Y) \in N. \tag{2.2.4}$$

The value of $C$ is adjusted so that $\int_N P_a(n)dn = 1$ (see [4, (2.6)]). For $\lambda \in \mathbb{C}$, we define $\wp_\lambda(x, n) = \wp(x, n)^{1/2 - i\lambda/Q} = \wp(x, n)^{-(i\lambda - \rho)/Q}$. Then it is known that for each fixed $n \in N$ $\Delta \wp_\lambda(x, n) = -(\lambda^2 + \rho^2)\wp_\lambda(x, n)$. The Poisson transform of a function $F$ on $N$ is defined as (see [4])

$$\mathfrak{P}_\lambda F(x) = \int_N F(n) \wp_\lambda(x, n) dn.$$

It follows that $\Delta \mathfrak{P}_\lambda F = -(\lambda^2 + \rho^2)\mathfrak{P}_\lambda F$. The elementary spherical function $\phi_\lambda(x)$ is given by

$$\phi_\lambda(x) = \int_N \wp_\lambda(x, n) \wp_{-\lambda}(e, n) dn.$$

It follows that $\phi_\lambda$ is a radial eigenfunction of $\Delta$ with eigenvalue $-(\lambda^2 + \rho^2)$ satisfying $\phi_\lambda(x) = \phi_{-\lambda}(x), \phi_\lambda(x) = \phi_\lambda(x^{-1})$ and $\phi_\lambda(e) = 1$. Since $\wp_{-i\rho}(x, n) \equiv 1$ for all $x \in S, n \in N$ and

$\wp_{i\rho}(x,n)=\wp(x,n)$, it follows that

$$\phi_{-i\rho}(x)=\int_N \wp_{i\rho}(e,n)=\int_N P_1(n)dn=1.$$

We have the following asymptotic estimate of $\phi_\lambda$ (see [3]). For $p\in(0,2]$, let $\gamma_p=2/p-1$. Then,

$$|\phi_{\alpha+i\gamma_p\rho}(x)|\asymp e^{-(2\rho/p')|x|},\quad \alpha\in\mathbb{R},0<p<2. \tag{2.2.5}$$

From this and (2.2.2) it follows that $\phi_\lambda\in L^{p',\infty}(S)$ (respectively $\phi_\lambda\in L^{p',1}(S)$) if and only if $|\Im\lambda|\le\gamma_p\rho$ (respectively $|\Im\lambda|<\gamma_p\rho$) for $1<p<2$ (see [35] for more details). The estimate above degenerates when $p=2$, i.e. when $\gamma_p=0$ and in this case we have $\phi_0(x)\asymp(1+|x|)e^{-\rho|x|}$. If $\lambda\in\mathbb{R}^\times$ and $t\ge 1$ then the Harish-Chandra series for $\phi_\lambda$ implies,

$$\phi_\lambda(a_t)=e^{-\rho t}[\mathrm{c}(\lambda)e^{i\lambda t}+c(-\lambda)e^{-i\lambda t}+E(\lambda,t)],\ \text{where } |E(\lambda,t)|\le C_\lambda e^{-2t}. \tag{2.2.6}$$

See [23, (3.11)]) for a proof of the above for the symmetric spaces. The proof works *mutatis mutandis* for general Damek-Ricci spaces. From this estimate it follows that $\phi_\lambda\in L^{2,\infty}(S)$ for any $\lambda\in\mathbb{R}^\times$.

We define the spherical Fourier transform $\widehat{f}$ of a suitable radial function $f$ as

$$\widehat{f}(\lambda)=\int_S f(x)\phi_\lambda(x)dx,$$

whenever the integral converges. For $1\le p\le 2$, the $L^p$-Schwartz space $C^p(S)$ is defined (see [3, 11]) as the set of $C^\infty$-functions on $S$ such that

$$\gamma_{r,D}(f)=\sup_{x\in S}|Df(x)|\phi_0^{-2/p}(1+|x|)^r<\infty,$$

for all nonnegative integers $r$ and left invariant differential operators $D$ on $S$. Let $C^p(S)^\#$ be the set of radial functions in $C^p(S)$. We define the strip $S_p=\{z\in\mathbb{C}\mid|\Im z|\le\gamma_p\rho\}$ where $\gamma_p=2/p-1$. Let $S_p^\circ$ and $\partial S_p$ respectively be the interior and the boundary of the strip and $C^p(\widehat{S})^\#$ be the set of even holomorphic functions on $S_p^\circ$ which are continuous on $\partial S_p$ and satisfy for all nonnegative integers $l,m$,

$$\nu_{l,m}(f)=\sup_{\lambda\in S_p}|\frac{d^l}{d\lambda^l}f(x)|(1+|\lambda|)^m<\infty.$$

When $p=2$ then the strip degenerates to the line $\mathbb{R}$ and $C^2(\widehat{S})^\#$ is defined as the set of even Schwartz class functions on $\mathbb{R}$. We topologize $C^p(S)$ and $C^p(\widehat{S})^\#$ by the seminorms $\gamma_{r,D}$ and by $\nu_{l,m}$ respectively. It is known that (see [3, 11]) $f\mapsto\widehat{f}$ is a topological isomorphism from $C^p(S)^\#$ to $C^p(\widehat{S})^\#$ for $1\le p\le 2$.

Apart from the weak $L^p$-norm, we need some size estimates which are close to $L^p$. Let $B(0,R)=\{x\in S\mid |x|<R\}$ be the geodesic ball of radius $R$. For a function $u$ on $S$ and $1<p<\infty,1\le q<\infty$ we define,

$$M_p(u) = \left(\limsup_{R\to\infty}\frac{1}{R}\int_{B(0,R)}|u(x)|^p dx\right)^{1/p}, \tag{2.2.7}$$

$$\mathcal{A}_{p,q}(u) = \|\mathcal{A}_q(u)\|_{p,\infty},\ \text{where } \mathcal{A}_q(u)(x)=\left(\int_{\partial B(\mathfrak{s})}|u(r\omega)|^q d\omega\right)^{1/q}. \tag{2.2.8}$$

2.3. **Symmetric spaces.** We recall that a rank one Riemannian symmetric space of noncompact type $X$ can be realized as the quotient space $G/K$ where $G$ is a connected noncompact semisimple Lie group with finite center and of real rank one and $K$ is a maximal compact subgroup of $G$. We consider the Iwasawa decomposition $G=NAK$. Then the subgroup $N$ is a $H$-type group. Therefore a rank one Riemannian symmetric space $X=G/K$ can be identified as $NA$ through this decomposition and the

space $X$ accommodates itself inside the DR spaces as an "Iwasawa $NA$ group". The $G$-invariant measure $dx$ on $X$ coincides with the left invariant Haar measure on $X$ viewed as a $NA$ group. The canonical Riemannian structure on $X$ as $NA$ group also coincides with the Riemannian structure induced by the Killing form, precisely $\langle Y_1, Y_2\rangle = -B(Y_1, \theta Y_2)$ where $B$ and $\theta$ respectively are the Killing form and Cartan involution of $\mathfrak{g}$, the Lie algebra of $G$ and $Y_1, Y_2 \in \mathfrak{g}$. A function on $X$ can be identified with a function on $G$ which is invariant under right $K$-action. Through this identification semisimple machineries can be brought forward to $X$, which we shall mention below.

The group $G$ (and in particular its subgroup $K$) acts naturally on $X$ by left translations. Let $M$ be the centralizer of $A$ in $K$. Apart from the Iwasawa decomposition $G = NAK$ mentioned above, we shall use the Iwasawa decomposition $G = KAN$. Through the action of $A$ on $N$ mentioned in the previous subsection (in other words since $A$ normalizes $N$) $G$ admits decompositions $G = KNA$ and $G = ANK$. It also has the polar decomposition $G = K\overline{A^+}K$, where $\overline{A^+}$ is identified with nonnegative real numbers. Using the Iwasawa decomposition $G = KAN$, we write an element $x \in G$ uniquely as $k(x) \exp H(x) n(x)$ where $k(x) \in K, n(x) \in N$ and $H(x) \in \mathfrak{a}$, where $\mathfrak{a}$ is the Lie algebra of $A$. Let $dg$, $dk$ and $dm$ be the Haar measures of $G$, $K$ and $M$ respectively with $\int_K dk = 1$ and $\int_M dm = 1$ and $dn$ be as given in subsection 2.2. We have the following integral formulae corresponding to the two Iwasawa decompositions $G = KAN$, $G = NAK$ and the polar decomposition, which hold for any integrable function:

(2.3.1)
$$\int_G f(g)dg = C_1 \int_K \int_{\mathbb{R}} \int_N f(ka_t n) e^{2\rho t}\, dn\, dt\, dk, \int_G f(g)dg = C_2 \int_K \int_{\mathbb{R}} \int_N f(na_t k) e^{-2\rho t}\, dn\, dt\, dk,$$

and

$$\int_G f(g)dg = C_3 \int_K \int_0^\infty \int_K f(k_1 a_t k_2)(\sinh t)^{m_\gamma} (\sinh 2t)^{m_{2\gamma}}\, dk_1\, dt\, dk_2, \tag{2.3.2}$$

where $m_\gamma$ and $m_{2\gamma}$ are the dimensions of the root spaces $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{2\gamma}$ respectively, $\gamma$ being the unique positive indivisible root and $\rho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})\gamma$, where $\gamma$ is treated as a positive number by $\gamma(1) = 1$. The constants $C_1, C_2, C_3$ depend on the normalization of the Haar measures involved. The formulae above are indeed coincides with (2.2.1) and (2.2.2) (with $\rho = Q$) when $G/K$ is treated as a $NA$ group. The apparent mismatch is due to the fact that in the former we take $\gamma(1) = 1/2$ instead of $\gamma(1) = 1$, to make our formulae consistent with the literature. As in the previous subsection for a function on $X$, we shall use the notation $J(t)$ to rewrite (2.3.2) as $\int_X f(x)dx = \int_K \int_0^\infty f(k_1 a_t) J(t) dt\, dk$.

We also note that using well known estimate $\sinh t \asymp te^t/(1+t), t \geq 0$ it follows from (2.3.2) that

$$\begin{aligned} \int_G |f(g)|dg &\asymp C_3 \int_K \int_0^1 \int_K |f(k_1 a_t k_2)| t^{d-1}\, dk_1\, dt\, dk_2 \\ &+ C_4 \int_K \int_1^\infty \int_K |f(k_1 a_t k_2)| e^{2\rho t}\, dk_1\, dt\, dk_2 \end{aligned} \tag{2.3.3}$$

where $d = m_\alpha + m_{2\alpha} + 1$.

It is well known that the *maximal distinguished boundary* (or boundary for short) of the symmetric space $X = G/K$ has two different, albeit essentially equivalent, realizations which we obtain through the Iwasawa decomposition of $G = KAN$. The compact boundary is $K/M$ and the noncompact one is the nilpotent group $N$. There is a natural correspondence between these two boundaries if we leave out an appropriate set of measure zero. If $G/K$ is realized as an Iwasawa $NA$ group, then as done in the previous subsection, we consider $N$ as the boundary and deal with the Poisson transform $\wp_\lambda$. We

shall define below the Poisson transform $\mathcal{P}_\lambda$ considering the compact boundary $K/M$. We refer to [4, pp. 418–419], for relation between $\mathcal{P}_\lambda$ and $\wp_\lambda$ defined in the previous subsection.

For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel: $x \mapsto e^{-(i\lambda+\rho)H(x^{-1})}$ is an eigenfunction of the Laplace Beltrami operator $\Delta$ with eigenvalue $-(\lambda^2+\rho^2)$. For any $\lambda \in \mathbb{C}$ and $F \in L^1(K/M)$ we define the Poisson transform $\mathcal{P}_\lambda$ of $F$ by (see [17, p. 279]) by

$$\mathcal{P}_\lambda F(x) = \int_{K/M} F(k) e^{-(i\lambda+\rho)H(x^{-1}k)} dk \text{ for } x \in X.$$

Then,

$$\Delta \mathcal{P}_\lambda F = -(\lambda^2+\rho^2)\mathcal{P}_\lambda F.$$

We recall that for these Iwasawa $NA$ groups a function is radial if and only if $f(kx) = f(x)$ for all $k \in K$ and $x \in X$. The radialization operator $R$ takes the simpler form: $Rf(x) = \int_K f(kx)dk$.

For any $\lambda \in \mathbb{C}$ the elementary spherical function $\phi_\lambda$ defined in subsection 2.2 has the following alternative expression,

$$\phi_\lambda(x) = \mathcal{P}_\lambda 1(x) = \int_{K/M} e^{-(i\lambda+\rho)H(xk)}\, dk \text{ for all } x \in G,$$

where by 1 we denote the constant function 1 on $K/M$ (see [4, 30]). It is clear that on $X$ the function (defined in subsection 2.2) $\mathcal{A}_q(u)(x) = \left(\int_{K/M} |u(kx)|^q dk\right)^{1/q}$.

## 3. Sharpness of Theorems A, B

In this section we shall try to motivate the formulation of the theorems stated in the introduction and establish their sharpness by answering the following natural questions:

(a) Does Theorem A hold true when $\alpha = 0$? Does Theorem B hold true when $\alpha = \beta \pm i\gamma_p\rho$ for $\beta \neq 0$?

(b) In Theorem A (respectively in Theorem B), is it possible to substitute $L^{2,\infty}$-norm (respectively $L^{p',\infty}$-norm) by any other Lorentz norms?

(c) Is it necessary to use both positive and negative integral powers of $\Delta$ in Theorem A ?

3.1. **Estimates of the Poisson transform.** We need to start with the basic $L^p$-behaviour of the Poisson transform as the Poisson transforms (of functions or functionals) form the set of eigenfunctions of the Laplacian. From the Kunze-Stein phenomenon and the Herz's principe de majoration (see [33, 7]) it follows that for $1 \le p < 2$, $p \le q \le p'$ and $\alpha \in \mathbb{R}$, the Poisson transform on $X$ satisfies the following estimates:

$$\|\mathcal{P}_{\alpha+i\gamma_q\rho} F\|_{p',\infty} \le C\|F\|_{L^{q'}(K/M)}. \tag{3.1.1}$$

Recently similar estimates for the Damek-Ricci spaces $S$ were also obtained by the authors of this paper (see [30, 35]): For $1 \le p < 2$, $p \le q \le p'$ and $\alpha \in \mathbb{R}$,

$$\|\mathfrak{P}_{\alpha+i\gamma_q\rho} F\|_{p',\infty} \le C\|F\|_{L^{q'}(N)}. \tag{3.1.2}$$

From the estimate of $\phi_0$ given in section 2, it follows that $\phi_0 \notin L^{2,\infty}(S)$ which obviates an analogue of (3.1.1) and (3.1.2) for $p = 2$. However, from (2.2.6), it follows that for real $\lambda \neq 0$, $\phi_\lambda \in L^{2,\infty}(S)$. Therefore one would expect an inequality on $X$ and $S$ respectively of the form:

$$\|\mathcal{P}_\lambda F\|_{2,\infty} \le C(\lambda)\|F\|_{L^2(K/M)}, \|\mathfrak{P}_\lambda F\|_{2,\infty} \le C(\lambda)\|F\|_{L^2(N)} \text{ for } \lambda \in \mathbb{R}^\times.$$

At this point of time such an inequality is not known. It is well known that there does not exists any eigenfunction of $\Delta$ which is in $L^p(S)$ for $p \leq 2$. But we have observed above that there are $L^{2,\infty}$-eigenfunctions of $\Delta$ (e.g. $\phi_\lambda$, $\lambda \in \mathbb{R}^\times$). We also have the following result.

**Proposition 3.1.1.** *Let $u$ be a nonzero function on $X$.*

(i) *If $\Delta u = -\rho^2 u$ then $u \notin L^{2,\infty}(X)$.*

(ii) *If $\Delta u = -(\lambda^2 + \rho^2)u$ for some $\lambda \in \mathbb{R}^\times$, then $u \notin L^{2,q}(X)$ for $q < \infty$.*

(iii) *If for some $1 < p < 2$, $\Delta u = -[(\beta \pm i\gamma_p\rho)^2 + \rho^2]u$ for $\beta \in \mathbb{R}$ then $u \notin L^{q',r}(X)$ if one of these two conditions is satisfied:* (a) $q > p$, (b) *if $q = p$ and $r < \infty$.*

*Proof.* We suppose that $\Delta u = -(\lambda^2 + \rho^2)u$ for some $\lambda \in \mathbb{C}$ and $u(x_0) \neq 0$ for some $x_0 \in X$. Then $f(y) = \int_K u(x_0 k y)dk$ satisfies $f(y) = \phi_\lambda(y)u(x_0)$ (see [18, p. 402]) and hence $f$ is a radial eigenfunction of $\Delta$ with the eigenvalue $-(\lambda^2 + \rho^2)$. Thus to prove (i) and (iii) respectively, it is enough to show that $\phi_0 \notin L^{2,\infty}(X)$ and $\phi_{\beta \pm i\gamma_p\rho} \notin L^{q',r}(X)$ for $q, r$ as in (iii), which are clear from the estimates of $\phi_0$ and $\phi_{\beta - i\gamma_p\rho}$ given in section 2. Similarly for (ii) it is enough to show that $\phi_\lambda$ with $\lambda \in \mathbb{R}^\times$ is not in $L^{2,q}(X)$ for any $q < \infty$, which we shall prove below.

We note that $\phi_\lambda \notin L^{2,q}(X)$ is equivalent to $\phi_\lambda \notin L^{2,q}(\mathbb{R}^+, J(t)dt)$ where $J(t)$ is the Jacobian in the polar decomposition (see (2.3.3)). Since $\phi_\lambda$ is a continuous function, and $J(t) \asymp e^{2\rho t}$ when $t \geq 1$, it suffices to show that for $t \geq 1$, $t \mapsto \phi_\lambda(a_t)$ is not in $L^{2,q}((1,\infty), e^{2\rho t}dt)$. Since, $E(\lambda, t) \leq C_\lambda e^{-2t}$ for $t \geq 1$ in (see (2.2.6)), the assumption that $\phi_\lambda(a_t)$ is in $L^{2,q}((1,\infty), e^{2\rho t}dt)$ will imply that $g(t) = e^{-\rho t}(\mathrm{c}(\lambda)e^{it\lambda} + \mathrm{c}(-\lambda)e^{-it\lambda}) = e^{-\rho t}2\Re(\mathrm{c}(\lambda)e^{it\lambda})$ is in $L^{2,q}((1,\infty), e^{2\rho t}dt)$. Let $\mathrm{c}(\lambda) = a(\lambda) + ib(\lambda)$. Then $g(t) = 2e^{-\rho t}(a(\lambda)\cos\lambda t - b(\lambda)\sin\lambda t)$. Since the translation operator $f(t) \mapsto f(t + \pi/2\lambda)$ is $(p,p)$ for any $p \geq 1$ in the measure space $((1,\infty), e^{2\rho t}dt)$, we get by interpolation (see [38, p. 197]) that the translation operator is bounded on $L^{2,q}(X)$. Thus we get $g(\bullet + \pi/2\lambda) \in L^{2,q}((1,\infty), e^{2\rho t}dt)$ and hence $b(\lambda)g(t) + a(\lambda)g(t + \pi/2\lambda) \in L^{2,q}((1,\infty), e^{2\rho t}dt)$. Since $g(t + \pi/2\lambda) = -2e^{-\rho t}(a(\lambda)\sin\lambda t + b(\lambda)\cos\lambda t)$, it follows that $b(\lambda)g(t) + a(\lambda)g(t + \pi/2\lambda) = -2e^{-\rho t}(b(\lambda)^2 + a(\lambda)^2)\sin\lambda t \in L^{2,q}((1,\infty), e^{2\rho t}dt)$, i.e. $e^{-\rho t}\sin\lambda t \in L^{2,q}((1,\infty), e^{2\rho t}dt)$. Similarly we can show that $e^{-\rho t}\cos\lambda t \in L^{2,q}((1,\infty), e^{2\rho t}dt)$. These two together imply that $e^{-\rho t} \in L^{2,q}((1,\infty), e^{2\rho t}dt)$, which is false as can be verified by direct computation. □

**Remark 3.1.2.** From (i) of the proposition above it is straightforward to see that the hypothesis of Theorem A with $\alpha = 0$ cannot yield any nonzero eigenfunction of $\Delta$. Similarly from (ii) of the proposition it follows that Theorem A with $L^{2,q}$-norm, $q < \infty$ replacing the $L^{2,\infty}$-norm will not have any nonzero solution either. In the same way (iii) discards the use of $L^{p',r}$-norm with $r < \infty$ and that of $L^{q',r}$-norm with $q > p$ in Theorem B. It is also noted above that $p < 2$ cannot be used in Theorem A and Theorem B. These along with the counter examples constructed in the next subsection will complete answering the questions.

3.2. **Counter examples.** Let $1 < p < 2$. We will show that if we substitute $i\gamma_{p'}\rho$ by $\beta \pm i\gamma_{p'}\rho$ in Theorem B with $\beta \in \mathbb{R}^\times$, then there exists a measurable function $f$ satisfying the hypothesis of the theorem but $f$ is not an eigenfunction of $\Delta$. We shall consider only $\alpha = \beta + i\gamma_p\rho$. The case $\alpha = \beta - i\gamma_p\rho$ will be analogous. To show this, we will appeal to the description of the $L^{p_1}$-spectrum of $\Delta$ for $1 \leq p_1 < \infty$. It is known that (see [32, 42, 3]) the $L^{p_1}$-spectrum $\sigma_{p_1}$ of $\Delta$ is the image of the set $S_{p_1}$ under the map $\Lambda(z) = -(z^2 + \rho^2)$. Precisely, $\sigma_{p_1}$ is given by the parabolic region

$$\sigma_{p_1} = \{-(z^2 + \rho^2) \mid |\Im z| \leq |\gamma_{p_1}\rho|\} = \sigma_{p_1'}. \tag{3.2.1}$$

We note that for $p$ as above if $p < q < 2$ then $\gamma_q < \gamma_p$, hence $\Lambda(\beta + i\gamma_q\rho) \in \sigma_p$ and $\phi_{\beta+i\gamma_q\rho} \in L^{p',1}(X) \subset L^{p',\infty}(X)$ (see section 2). Since $\beta \neq 0$ we can choose $r$ such that $p < q < r < 2$ and real numbers $s, t$ such that $|\Lambda(s + i\gamma_{q'}\rho)| = |\Lambda(t + i\gamma_{r'}\rho)| = |\Lambda(\beta + i\gamma_{p'}\rho)|$. Figure 1 below explains the situation.

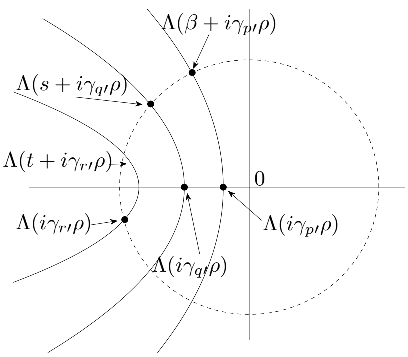


FIGURE 1. $1 < p < q < 2$

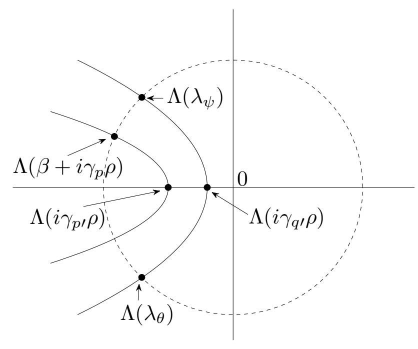


FIGURE 2. $1 < q < p < 2$

Hence there exists $\theta$ and $\psi$ in $(0, 2\pi)$ such that

$$\Lambda(s + i\gamma_{q'}\rho)e^{-i\theta} = \Lambda(t + i\gamma_{r'}\rho)e^{-i\psi} = \Lambda(\beta + i\gamma_{p'}\rho). \tag{3.2.2}$$

We define

$$f(x) = \phi_{t+i\gamma_{r'}\rho}(x) + \phi_{s+i\gamma_{q'}\rho}(x).$$

Using (3.2.2) it follows that

$$\begin{aligned}\Delta^k f(x) &= (\Lambda(t + i\gamma_{r'}\rho))^k \phi_{t+i\gamma_{r'}\rho} + (\Lambda(s + i\gamma_{q'}\rho))^k \phi_{s+i\gamma_{q'}\rho} \\ &= e^{ik\psi}(\Lambda(\beta + i\gamma_{p'}\rho))^k \phi_{t+i\gamma_{r'}\rho} + e^{ik\theta}(\Lambda(\beta + i\gamma_{p'}\rho)^k \phi_{s+i\gamma_{q'}\rho} \\ &= (\Lambda(\beta + i\gamma_{p'}\rho))^k f(x).\end{aligned}$$

Therefore $|\Delta^k f| \leq |\Lambda(\beta + i\gamma_p\rho)|^k(|\phi_{i\gamma_r\rho}| + |\phi_{i\gamma_q\rho}|$. Since both $\phi_{i\gamma_r\rho}$ and $\phi_{i\gamma_q\rho}$ are in $L^{p',\infty}(X)$ (see section 2), $f$ satisfies the hypothesis of Theorem B. It is clear that $f$ is not an eigenfunction of $\Delta$.

Next we shall show that in Theorem B, if we take $\alpha = \beta \pm i\gamma_p\rho$, $\beta \in \mathbb{R}$ and substitute $L^{p',\infty}$-norm by $L^{q',r}$-norm with $1 \leq q < p < 2$, then there are functions $f$ which satisfy the hypothesis, but they are not eigenfunctions of $\Delta$. As above we shall only consider the case $\alpha = \beta + i\gamma_p\rho$. Indeed (using the notation $\Lambda$ given above) there exists $\theta, \psi$ and $\lambda_\theta, \lambda_\psi \in S_q^\circ$ such that $\Lambda(\lambda_\psi)e^{-i\psi} = \Lambda(\lambda_\theta)e^{-i\theta} = \Lambda(\alpha) = -(\alpha^2 + \rho^2)$ (see Figure 2). As above we define $f = \phi_{\lambda_\theta} + \phi_{\lambda_\psi}$. Then

$$\Delta^k f = (\Lambda(\lambda_\theta))^k \phi_{\lambda_\theta} + (\Lambda(\lambda_\psi))^k \phi_{\lambda_\psi} = \Lambda(\alpha)^k [e^{ik\theta}\phi_{\lambda_\theta} + e^{ik\psi}\phi_{\lambda_\psi}].$$

From this it is clear that $f$ satisfies the hypothesis of Theorem B and $f$ is not an eigenfunction of $\Delta$. A similar construction will show that in Theorem A, if we substitute $L^{2,\infty}$-norm by $L^{p',r}$-norm with $2 < p' < \infty, 1 \leq r \leq \infty$ or by $L^\infty$-norm, then there exist functions which satisfy the hypothesis, despite not being eigenfunctions of $\Delta$. This completes answering the questions and thereby establishes the sharpness of Theorem A and Theorem B.

Lastly we shall show that unlike Theorem B, it is necessary to use positive as well as negative integral powers of $\Delta$ in Theorem A. Let $f = \phi_{\lambda_1} + \phi_{\lambda_2}$ for $\lambda_1, \lambda_2 \in \mathbb{R}^\times, \lambda_1 \neq \lambda_2$ and $(\lambda_i^2 + \rho^2)/(\alpha^2 + \rho^2) < 1$

for $i = 1, 2$. Then $\Delta^k f = -(\lambda_1^2 + \rho^2)^k \phi_{\lambda_1} - (\lambda_2^2 + \rho^2)^k \phi_{\lambda_2}$ and hence $|\Delta^k f| \leq (|\phi_{\lambda_1}| + |\phi_{\lambda_2}|)(\alpha^2 + \rho^2)^k$. Therefore $\|\Delta^k f\|_{2,\infty} \leq M(\alpha^2 + \rho^2)^k$ where $M = \|\phi_{\lambda_1}\|_{2,\infty} + \|\phi_{\lambda_2}\|_{2,\infty} < \infty$. But it is clear that $f$ is not an eigenfunction of $\Delta$.

## 4. Characterization of eigenfunction

In this section we shall focus mainly on Iwasawa $NA$ groups, in other words on the Riemannian symmetric spaces $X$ of noncompact type with real rank one. The purpose is to prove certain representation theorems for eigendistribution of the Laplace- Beltrami operator $\Delta$ on $X = G/K$. In particular we shall generalize Theorem 4.1.7 (b) for $p \in (1, 2)$ which was conjectured in [6] (see Theorem 4.3.6 below). To put things in proper perspective we shall first look at the available results. However being comprehensible on this vast topic is beyond the scope of this article. Instead, we shall restrict our attention only on those results which are related to the main theme of this paper. Throughout this section for a complex number $\lambda$,

$$E_\lambda = \{u \in C^\infty(X) \mid \Delta u = -(\lambda^2 + \rho^2)u\}.$$

4.1. **A brief Survey.** We have the following theorem on $X$ which can be viewed as an analogue of a result ([38, p. 50, Theorem 2.5]) on the upper half plane $\mathbb{R}^+_{n+1} = \{(x, y) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n, y > 0\}$.

**Theorem 4.1.1** (Furstenberg [12], Knapp-Williamson [28])**.** *A harmonic function $u : X \to \mathbb{C}$ satisfies*

$$\sup_{t>0} \left( \int_{K/M} |u(ka_t)|^p dk \right)^{1/p} < \infty, 1 < p < \infty, \quad (\textit{respectively, } \|u\|_\infty < \infty)$$

*if and only if there exists $f \in L^p(K/M)$ (respectively $f \in L^\infty(K/M)$) such that $u = \mathcal{P}_{-i\rho} f$.*

See also [39]. An analogue of the case $p = \infty$ of the result above for general $NA$ groups was proved by Damek in [8]. It is known that the theorem above is not true for bounded eigenfunctions of $\Delta$ with nonzero eigenvalues (see [14]).

Going towards the representation of eigenfunctions of $\Delta$ on $X$ with nonzero eigenvalue, we recall that an eigendistribution $u$ of $\Delta$ on $X$ is a real analytic function and is Poisson transform of an *analytic functional* $T$ defined on $K/M$ (see [15], [16], [25]). Lewis [31] puts additional condition on $u$ to ensure that the analytic functional $T$ is a distribution on $K/M$.

**Theorem 4.1.2** (Lewis)**.** *If $u$ is an eigenfunction of $\Delta$ then $u$ is Poisson transform of a distribution $T$ defined on $K/M$ if and only if there exists $\beta > 0$ such that $|u(ka_t)| \leq Ce^{\beta t}$ for all $k \in K/M$ and $t > 0$.*

We also have ([5, Theorem 3.2 (v)]),

**Lemma 4.1.3.** *Let $u \in E_\lambda(X)$ with $\Im\lambda < 0$ or $\lambda = 0$. If $u$ is a Poisson transform of a distribution $T$ defined on $K/M$, then $u(ka_t)/\phi_\lambda(a_t)$ converges to $T$ in the sense of distribution as $t \to \infty$.*

Sjögren ([37, Theorem 6.1]) determines the size estimates on $u$ which are sufficient to imply that the distribution $T$ is actually given by a function.

**Theorem 4.1.4** (Sjögren, [37])**.** *Let $u \in E_\lambda$ for some $\lambda \in \mathbb{C}$ with $\Im\lambda < 0$ or $\lambda = 0$. For $1 < p \leq \infty$, and $\beta > 0$, the function $ka_t \mapsto \phi_{\Im\lambda}(a_t)^{-1} e^{-\beta t/p} u(ka_t)$ belongs to $L^{p,\infty}(X, m_\beta)$ if and only if $u = \mathcal{P}_\lambda f$ for some $f \in L^p(K/M)$. Here $dm_\beta(x) = dm_\beta(ka_t) = e^{(\beta - 2\rho)t} J(t) dk dt$, $t > 0$, $k \in K/M$.*

Taking $\beta = 2\rho$ and $p = q'$, $1 \leq q < 2$ in Theorem 4.1.4 we get an $L^p$-version of the result of Furstenberg mentioned above.

**Corollary 4.1.5.** *Let $1 \leq q < 2$ and $\lambda = \alpha - i\gamma_q\rho$, $\alpha \in \mathbb{R}$ and $u \in E_\lambda$. Then $u \in L^{q',\infty}(X)$ if and only if $u = \mathcal{P}_\lambda f$ for some $f \in L^{q'}(K/M)$. In particular, if $q = 1$ then bounded eigenfunctions of $\Delta$ with eigenvalue $\alpha(2i\rho - \alpha)$ are Poisson transform of bounded functions on $K/M$.*

The case $1 < q < 2, \alpha = 0$ of the result above was proved independently in [33]. The following result in [5] generalizes Theorem 4.1.1 for eigenfunctions of $\Delta$ with nonzero eigenvalues.

**Theorem 4.1.6** (Ben Saïd et. al.)**.** *Let $\Im\lambda < 0$ or $\lambda = 0$, $1 < p \leq \infty$ and $u \in E_\lambda$. Then $u = \mathcal{P}_\lambda f$ for some $f \in L^p(K/M)$ if and only if $\sup_{t>0} \phi_{\Im\lambda}(a_t)^{-1} \left(\int_{K/M} |u(ka_t)|^p dk\right)^{1/p} < \infty$ (with usual modification for $L^\infty$-norm). If $p = 1$ then $u = \mathcal{P}_\lambda \mu$ for some signed measure $\mu$ on $K/M$.*

We observe that all these results leave out the case $\lambda \in \mathbb{R}^\times$. Motivated by a work of Strichartz ([40]) this was taken up by Ionescu and Boussejra et al. ([23, 6]) and their works reveal that the oscillatory behaviour of $\phi_\lambda, \lambda \in \mathbb{R}^\times$ plays a critical role in this case.

**Theorem 4.1.7** (Ionescu, Boussejra et al.)**.** *Suppose that $u \in E_\lambda$ with $\lambda \in \mathbb{R}^\times$.*

(a) *Then $u = \mathcal{P}_\lambda f$ for some $f \in L^2(K/M)$ if and only if $M_2(u) < \infty$. Moreover, in this case*
$$M_2(u) = |\mathrm{c}(\lambda)| \|f\|_{L^2(K/M)} \text{ for all } f \in L^2(K/M).$$

(b) *For $p \geq 2$ and $X$ a hyperbolic space over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $u = \mathcal{P}_\lambda f$ for some $f \in L^p(K/M)$ if and only if $\sup_{t>0} e^{\rho t} \mathcal{A}_p(u)(a_t) < \infty$.*

It was conjectured in [6] that (b) holds also for all $p \in (1, 2)$, which we shall prove in Theorem 4.3.6.

## 4.2. **Preparatory Lemmas.**

**Lemma 4.2.1.** *Let $u \in E_\lambda$, with $\lambda \in \mathbb{C}$ and $1 < p < \infty, 1 \leq q < \infty$. We denote by $\|\cdot\|$ any of these three norms $\|\cdot\|_{p,\infty}$, $M_p$ or $\mathcal{A}_{p,q}$. If $\|u\| < \infty$ then there exists $\alpha > 0$, $C > 0$, both of which may depend on $u$ such that*

$$|u(ka_s)| \leq C\|u\| e^{\alpha s} \text{ for all } s > 0 \tag{4.2.1}$$

*Proof.* Since $u$ is continuous it is enough to assume that $s$ is large. We shall first take the case $\|u\|_{p,\infty} < \infty$. We consider the ball $B(0, e^{-s}) = \{k_1 a_r \mid k_1 \in K, 0 < r < e^{-s}\}$, $s > 1$ and using polar coordinates we get

$$|B(0, e^{-s})| \asymp \int_0^{e^{-s}} r^{d-1} dr = C_d e^{-ds}, \tag{4.2.2}$$

where $d = \dim X$. By generalized mean value property of eigenfunctions of the Laplacian ([17], p.402) we have

$$u(g)\phi_\lambda(x) = \int_K u(gkx) dk,\ g \in G, x \in X. \tag{4.2.3}$$

Since $\phi_\lambda(e) = 1$ it follows that for large $s$, we have $|\phi_\lambda(x)| \geq C$ for all $x \in B(0, e^{-s})$. Therefore from (4.2.3) we conclude that

$$|u(g)| \leq C \int_K |u(gkx)| dk, \text{ for all } x \in B(0, e^{-s}) g = k_1 a_s. \tag{4.2.4}$$

Integrating both sides of (4.2.4) over $B(0,e^{-s})$ and noting that $B(0,e^{-s})$ is $K$-invariant, we get

$$\begin{aligned} |B(0,e^{-s})||u(g)| &\leq C\int_{B(0,e^{-s})}\int_K |u(gkx)|dkdx \\ &= C\int_{B(0,e^{-s})}|u(gx)|dx \\ &\leq C\|u\|_{L^{p,\infty}(X)}\|\chi_{B(0,e^{-s})}\|_{L^{p,1}(X)} \\ &= C\|u\|_{L^{p,\infty}(X)}|B(0,e^{-s})|^{1/p}. \end{aligned}$$

From above and (4.2.2) we get for large $s$

$$|u(g)| = |u(k_1a_s)| \leq C\|u\|_{L^{p,\infty}(X)}e^{\frac{d}{p'}s}.$$

This completes the proof when $\|u\|_{p,\infty}<\infty$.

Now we assume that $M_p(u)<\infty$. As above in this case also it follows that if $g=k_1a_s$ then

$$|B(0,e^{-s})|u(g)| \leq C\int_{B(0,e^{-s})}|u(gx)|dx \leq C\int_{B(0,e^{-s}+s)}|u(x)|dx, \tag{4.2.5}$$

as $d(0,x)\leq e^{-s}$ implies $d(0,gx)\leq d(0,g)+d(g,gx)\leq s+e^{-s}$ and consequently $gB(0,e^{-s})\subset B(0,e^{-s}+s)$. By Hölder's inequality from (4.2.5), we get

$$|B(0,e^{-s})|u(ka_s)| \leq \left(\int_{B(0,e^{-s}+s)}|u(x)|^pdx\right)^{1/p}|B(0,e^{-s}+s)|^{1/p'}. \tag{4.2.6}$$

Since $|B(0,e^{-s})| = e^{-ds}$ and for large $s$, $|B(0,e^{-s}+s)|\leq e^{4\rho s}$, it follows from (4.2.6) that

$$|u(ka_s)| \leq ce^{(d+\frac{1}{p})s}e^{\frac{4\rho s}{p'}}M_p(u), \text{ for all large } s.$$

Lastly we assume that $\mathcal{A}_{p,q}(u)<\infty$. From (4.2.5) it follows that for all large $s$

$$|B(0,e^{-s})|u(g)| \leq C\int_{B(0,e^{-s}+s)}|u(x)|dx \leq C\int_{B(0,2s)}|u(x)|dx \leq C\int_0^{2s}\int_K|u(ka_t)|J(t)dtdk \tag{4.2.7}$$

Hence from (4.2.7) we get,

$$|B(0,e^{-s})|u(g)| \leq \int_0^{2s}\left(\int_K|u(ka_t)|^qdk\right)^{1/q}J(t)dt \leq \mathcal{A}_{p,q}(u)|B(0,2s)|^{1/p'}. \tag{4.2.8}$$

Since for large $s$, $|B(0,e^{2s})|\asymp Ce^{4\rho s}$ it follows from (4.2.2) and (4.2.8) that

$$|u(ka_s)| \leq Ce^{ds}e^{\frac{4\rho s}{p'}}\mathcal{A}_{p,q}(u) \leq C\mathcal{A}_{p,q}(u)e^{\alpha s} \text{ for all large } s.$$

This completes the proof. $\square$

The next lemma compares $M_p(u)$ and $\|u\|_{p,\infty}$ when $u$ is an eigenfunctions of $\Delta$. The technique can be traced back to [37].

**Lemma 4.2.2.** *If $u\in C(X)\cap L^{p,\infty}(X)$, $1<p<\infty$ and there exist $\alpha>0$, $C>0$ such that for all $t>0$, and $k\in K$, $|u(ka_t)|\leq C\|u\|_{L^{p,\infty}(X)}e^{\alpha t}$ then for all $R>0$,*

$$\int_{B(0,R)}|u(x)|^pdx \leq C_{\alpha,p}\|u\|_{L^{p,\infty}(X)}R, \quad \textit{equivalently} \quad M_p(u)\leq C_{\alpha,p}\|u\|_{L^{p,\infty}(X)}. \tag{4.2.9}$$

*Proof.* Let $u^*$ be the decreasing rearrangement of $u$. Then it follows from the definition that

$$u^*(s)^p \leq \frac{1}{s}\|u\|_{p,\infty}^p, \text{ for all } s > 0. \tag{4.2.10}$$

We also have ([13, p. 64, Proposition 1.4.5, (7), (11)])

$$\int_{B(0,R)} |u(x)|^p dx \leq c \int_0^\infty (\chi_{B(0,R)}|u|)^*(t)^p dt. \tag{4.2.11}$$

Since $u$ grows at most exponentially, we have,

$$|\chi_{B(0,R)}(x)u(x)| \leq \|u\|_{L^{p,\infty}(X)} e^{\alpha R}\chi_{B(0,R)}(x). \tag{4.2.12}$$

As $(\chi_{B(0,R)})^* = \chi_{(0,|B(0,R)|)}$ it follows from (4.2.12) that ([13], Proposition 1.4.5, (3))

$$(\chi_{B(0,R)}|u|)^*(t)^p \leq C\|u\|_{p,\infty}^p e^{\alpha p R}\chi_{(0,|B(0,R)|)}(t). \tag{4.2.13}$$

When $R > 1$, we have from (4.2.10), (4.2.11) and (4.2.13)

$$\begin{aligned}
\int_{B(0,R)} |u(x)|^p dx &\leq C \int_0^{|B(0,R)|} \min\left\{\|u\|_{p,\infty}^p e^{p\alpha R}, \|u\|_{p,\infty}^p \frac{1}{t}\right\} dt \\
&\leq C\|u\|_{p,\infty}^p \int_0^{e^{2\rho r}} \min\left\{e^{p\alpha R}, \frac{1}{t}\right\} dt \\
&= C\|u\|_{p,\infty}^p \left(\int_0^{e^{-p\alpha R}} e^{p\alpha R} dt + \int_{e^{-p\alpha R}}^{e^{2\rho r}} \frac{dt}{t}\right) \\
&= C\|u\|_{p,\infty}^p (1 + 2\rho R + p\alpha R) \\
&\leq C_p \|u\|_{p,\infty}^p R. \qquad (4.2.14)
\end{aligned}$$

When $R \leq 1$, $|B(0,R)| \asymp R^{d-1}$ and for all $t \in [0, R^{d-1}]$ we have,

$$\min\{e^{p\alpha R}, \frac{1}{t}\} \leq Ce^{p\alpha R} \leq Ce^{p\alpha}.$$

Hence

$$\int_0^{|B(0,R)|} \min\{e^{p\alpha R}, \frac{1}{t}\} dt \leq C_{\alpha,p} \int_0^{R^{d-1}} e^{p\alpha} dt \leq C_{\alpha,p} R, \tag{4.2.15}$$

as $d \geq 2$. Combining (4.2.14) and (4.2.15) it follows that

$$\int_{B(0,R)} |u(x)|^p dx \leq C_{\alpha,p}\|u\|_{p,\infty}^p R, \text{ for all } R > 0. \qquad \square$$

4.3. **Main results on characterization.** We need the following results.
(I) Lt $\lambda = \alpha + i\gamma_{p'}\rho$, $1 < p < 2$, $\alpha \in \mathbb{R}$. For $t > 0$ and a measurable function $f$ on $K/M$ we define the maximal function

$$\widetilde{f}(k) = \sup_{t>0} \phi_{i\gamma_p\rho}(a_t)^{-1}(e^{-(i\lambda+\rho)H(a_t^{-1}\cdot)} * f)(k)$$

where the convolution is on $K$. From this we have the pointwise estimate

$$\mathcal{P}_{\alpha+i\gamma_{p'}\rho} f(ka_t)| \leq C\phi_{i\gamma_p\rho}(a_t)\widetilde{f}(k). \tag{4.3.1}$$

Michelson ([34]) proved that $\widetilde{f}$ satisfies, $\|\widetilde{f}\|_{L^r(K/M)} \leq C\|f\|_{L^r(K/M)}, 1 < r < \infty$. (see also [33]).
(II) For a nonnegative continuous function $\Phi$ defined on $[1, \infty)$ if there exists a constant $C > 0$ such that $\int_1^R \Phi(t)dt \leq CR$, for all $R > 1$ then (see [37])

$$\liminf_{t\to\infty} \Phi(t) < \infty. \tag{4.3.2}$$

The next result can be considered as an $L^p$ version of Theorem 4.1.7 (a).

**Theorem 4.3.1.** *Suppose that $1 < p < 2$ and $u \in E_\lambda$ for some $\lambda = \alpha + i\gamma_{p'}\rho$, $\alpha \in \mathbb{R}$. If $M_{p'}(u) < \infty$ then $u = \mathcal{P}_\lambda f$ for some $f \in L^{p'}(K/M)$. Moreover $M_{p'}(u) \asymp \|f\|_{L^{p'}(K/M)}$.*

*Proof.* We suppose that $M_{p'}(u) < \infty$. It then follow that there exists $R_0 > 1$ such that for all $R \geq R_0$,

$$\int_1^R \int_K |u(ka_t)|^{p'} dk e^{2\rho t} dt \leq C M_{p'}(u)^{p'} R.$$

That is,

$$\int_1^R \int_K \left| \frac{u(ka_t)}{\phi_\lambda(a_t)} \right|^{p'} dk dt \asymp \int_1^R \int_K \left| \frac{u(ka_t)}{e^{-2\rho t/p'}} \right|^{p'} dk dt \leq C M_{p'}(u)^{p'} R. \tag{4.3.3}$$

By (4.3.2) there exists a sequence $\{t_j\} \to \infty$ such that

$$\lim_{t_j \to \infty} \int_K \left| \frac{u(ka_{t_j})}{\phi_\lambda(a_{t_j})} \right|^{p'} dk \leq C M_{p'}(u)^{p'}. \tag{4.3.4}$$

This, in particular, implies that the sequence of functions $\left\{ \frac{u(\cdot\ a_{t_j})}{\phi_\lambda(a_{t_j})} \right\}, j = 1, 2 \ldots, \infty$ is a norm bounded family in $L^{p'}(K/M)$. By Eberlein-Šmulian theorem there exists a subsequence of $\{t_j\}$ (which we will continue to call $\{t_j\}$) and an $f \in L^{p'}(K)$ such that

$$\lim_{t_j \to \infty} \int_K \frac{u(ka_{t_j})}{\phi_\lambda(a_{t_j})} \psi(k) dk = \int_k f(k)\psi(k) dk \text{ for all } \psi \in L^p(K).. \tag{4.3.5}$$

It also follow from (4.3.4) that $\|f\|_{L^{p'}(K/M)} \leq C M_{p'}(u)$. On the other hand, by Lemma 4.2.1 and Theorem 4.1.2, we know that $u = \mathcal{P}_\lambda T$ for some distribution $T$ on $K/M$. It then follows from Lemma 4.1.3 that

$$\lim_{t \to \infty} \int_K \frac{u(ka_t)}{\phi_\lambda(a_t)} \psi(k) dk = T(\psi) \text{ for all } \psi \in C^\infty(K).$$

In particular,

$$\lim_{t \to \infty} \int_K \frac{u(ka_{t_j})}{\phi_\lambda(a_{t_j})} \psi(k) dk = T(\psi) \text{ for all } \psi \in C^\infty(K) \tag{4.3.6}$$

From (4.3.5) and (4.3.6) now we have

$$T(\psi) = \int_K f(k)\psi(k) dk, \text{ for all } \psi \in C^\infty(K)$$

i. e., $T = f$. It now follows easily from (4.3.1) that $M_{p'}(u) = M_{p'}(\mathcal{P}_\lambda f) \leq C\|f\|_{L^{p'}(K/M)}$. □

**Remark 4.3.2.** If $u \in E_\lambda \cap L^{p',\infty}(X)$, for $\lambda$ and $p$ are as in Theorem 4.3.1, then by Corolary 4.1.5, $u = \mathcal{P}_\lambda f$ for some $f \in L^{p'}(K/M)$. Therefore by Lemma 4.2.2 and Theorem 4.3.1 $\|f\|_{L^{p'}(K/M)} \leq C\|u\|_{p',\infty}$. This along with (3.1.1) shows $\|u\|_{p',\infty} \asymp \|f\|_{L^{p'}(K/M)}$.

We offer another characterization of eigenfunctions which generalizes Theorem 4.1.6 for $-\rho < \Im\lambda < 0$.

**Theorem 4.3.3.** *Let $1 < p < 2$, $1 < q < \infty$ and $u \in E_\lambda$ with $\lambda = \alpha + i\gamma_{p'}\rho$. Then $u = \mathcal{P}_\lambda f$ for some $f \in L^q(K/M)$ if and only if $\mathcal{A}_{p',q}(u) < \infty$.*

*Proof.* We assume that $\mathcal{A}_{p',q}(u) < \infty$, i.e.,

$$s\mathcal{A}_q(u)^*(s)^{p'} \leq \mathcal{A}_{p',q}(u)^{p'}, \text{ for all } s > 0. \tag{4.3.7}$$

By Lemma 4.2.1, we have that for all $k \in K$ and $t > 0$,

$$|u(ka_t)| \leq C\mathcal{A}_{p',q}(u) e^{\alpha t}$$

which in particular implies by Theorem 4.1.2 that $u = \mathcal{P}_\lambda T$ for some distribution $T$ on $K/M$. Taking $L^p$-norm on $K$ of both sides of the inequality above we get for all $t > 0$

$$\mathcal{A}_q(u)(a_t) \leq C\mathcal{A}_{p',q}(u)e^{\alpha t}. \tag{4.3.8}$$

Using estimate of $\phi_{i\gamma_p\rho}$ (see section 2) we have for all $R > 1$

$$\begin{aligned} \int_1^R \left(\int_K \left|\frac{u(ka_t)}{\phi_{i\gamma_{p'}\rho(a_t)}}\right|^q dk\right)^{p'/q} dt &\asymp \int_{E_R} \mathcal{A}_q(u)(x)^{p'} dx \\ &\leq C\int_0^\infty (\chi_{E_R}\mathcal{A}_q(u))^*(s)^{p'} ds, \end{aligned} \tag{4.3.9}$$

where $E_R = \{x \in X \mid 1 \leq |x| \leq R\}$. As $\chi^*_{E_R} = \chi_{[0,|E_R|)}$ it follows from (4.3.8) that

$$(\chi_{E_R}\mathcal{A}_q(u))^*(s)^{p'} \leq C\mathcal{A}_{p',q}(u)^{p'} e^{p'\alpha R}\chi_{[0,|E_R|)}(s), \text{ for all } s > 0. \tag{4.3.10}$$

Hence, as in the proof of Theorem 4.3.1, it follows from (4.3.9), (4.3.10) and (4.3.7) that

$$\begin{aligned} \int_1^R \left(\int_K \left|\frac{u(ka_t)}{\phi_{i\gamma_{p'}\rho(a_t)}}\right|^q dk\right)^{p'/q} dt &\leq C\mathcal{A}_{p',q}(u)^{p'} \int_0^{|E_R|} \min\left\{\frac{1}{t}, e^{p'\alpha R}\right\} dt \\ &\leq C_{\alpha,p}\mathcal{A}_{p',q}(u)^{p'} R \end{aligned} \tag{4.3.11}$$

By (4.3.2) there is a sequence $\{t_j\} \to \infty$ such that,

$$\lim_{j\to\infty} \left(\int_K \left|\frac{u(ka_t)}{\phi_{i\gamma_{p'}\rho(a_t)}}\right|^q dk\right)^{1/q} \leq C\mathcal{A}_{p',q}(u).$$

Hence by Lemma 4.1.3 we have

$$\lim_{j\to\infty} \int_K \frac{u(ka_t)}{\phi_{i\gamma_p\rho}(a_t)}\psi(k)dk = T(\psi), \text{ for all } \psi \in C^\infty(K/M) \tag{4.3.12}$$

for some distribution $T$ on $K/M$. As in Theorem 4.3.1, an application of Eberlein-Šmulian theorem shows that there exists $f \in L^q(K/M)$ such that $u = \mathcal{P}_\lambda f$ and

$$\|f\|_{L^q(K/M)} \leq C\mathcal{A}_{p',q}(u).$$

Using (4.3.1) and the estimate of the maximal function $\widetilde{f}$ mentioned there we have,

$$\mathcal{A}_q(u)(a_t) = \left(\int_{K/M} \left|\mathcal{P}_{i\gamma_{p'}\rho}f(ka_t)\right|^q\right)^{1/q} \leq C\|f\|_{L^q(K/M)}\phi_{i\gamma_{p'}\rho}(a_t).$$

The estimate of $\phi_{i\gamma_p\rho}$ (see section 2) then shows that $\mathcal{A}_q(u) \in L^{p',\infty}(X)$. □

**Remark 4.3.4.** It is clear that for $1 < p < 2$, $1 < q < \infty$ and $\lambda = \alpha - i\gamma_p\rho$, $\alpha \in \mathbb{R}$, there exists a positive constant $C$ such that for all $u$ we have $\mathcal{A}_{p',q}(u) \leq C\sup_{\{a_t|t>0\}} \phi_{i\gamma_p\rho}(a_t)^{-1}\mathcal{A}_q(u)(a_t)$. Thus it follows that Theorem 4.3.3 generalizes Theorem 4.1.7 for these values of $\lambda$.

In the rest of the section we shall consider only $\lambda \in \mathbb{R}^\times$. We know from Corollary 4.1.5 that if $1 < p < 2$ and $\Im\lambda = \gamma_{p'}\rho$ then $\mathcal{P}_\lambda f \in L^{p',\infty}(X)$ whenever $f \in L^{p'}(K/M)$. It is not known to us whether $\lambda \in \mathbb{R}^\times$ and $f \in L^2(K/M)$ ensure that the Poisson transform $\mathcal{P}_\lambda f \in L^{2,\infty}(X)$. However the following result shows that the converse is true.

**Theorem 4.3.5.** *If $u \in E_\lambda \cap L^{2,\infty}(X)$ for some $\lambda \in \mathbb{R}^\times$ then $u = \mathcal{P}_\lambda f$ for some $f \in L^2(K/M)$.*

*Proof.* By Lemma 4.2.1 and Lemma 4.2.2 it follows that $M_2(u) \leq C\|u\|_{L^{2,\infty}(X)}$. Applying Theorem 4.1.7 (a), we conclude that $u = \mathcal{P}_\lambda f$ for some $f \in L^2(K/M)$. $\square$

The last result of this section will be restricted to the hyperbolic spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We will need the following results proved in [6, Theorem B and Theorem A, (ii)]:
**(I)** If $1 < p < \infty$ and $\lambda \in \mathbb{R}^\times$ then there exists a positive constant $C$ such that for all $f \in L^p(K/M)$, the following estimate holds,

$$\sup_{t>0} e^{\rho t} \left( \int_{K/M} |\mathcal{P}_\lambda f(ka_t)|^p dk \right)^{1/p} \leq C_p \|f\|_{L^p(K/M)}. \tag{4.3.13}$$

**(II)** Let $f \in L^2(K/M)$ and $\lambda \in \mathbb{R}^\times$. We define $f_R(b) = \frac{1}{R} |\mathrm{c}(\lambda)|^{-2} \int_{B(0,R)} \mathcal{P}_\lambda f(x) e^{(i\lambda-\rho)H(x^{-1}b)} dx$ for $b \in K/M$. Then

$$f_R \longrightarrow f \text{ in } L^2(K/M) \text{ as } R \to \infty. \tag{4.3.14}$$

**Theorem 4.3.6.** *Let $1 < q < \infty$ and $X$ be hyperbolic space over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. If $u \in E_\lambda$ with $\lambda \in \mathbb{R}^\times$ then $u = \mathcal{P}_\lambda f$, for some $f \in L^q(K/M)$ if and only if $\mathcal{A}_{2,q}(u) < \infty$ (consequently if and only if $\sup_{t>0} e^{\rho t} \mathcal{A}_q(u) < \infty$).*

This theorem is slightly general than the conjecture posed in [6] (see the line following Theorem 4.1.7).

*Proof.* If $u = \mathcal{P}_\lambda f$ for some $f \in L^q(K/M)$ then by (4.3.13) $\sup_{t>0} e^{\rho t} \mathcal{A}_q(u)(a_t) < \infty$, consequently $\mathcal{A}_{2,q}(u) \leq C \sup_{t>0} e^{\rho t} \mathcal{A}_q(u) < \infty$. Therefore to complete the proof it is enough to show that $\mathcal{A}_{2,q}(u) < \infty$ implies $u = \mathcal{P}_\lambda f$ for some $f \in L^q(K/M)$.

We first consider the case $q = 2$, that is, $u \in E_\lambda$ and $\mathcal{A}_{2,2}(u) < \infty$. We note that

$$\int_{B(0,R)} |u(x)|^2 dx = \int_0^R \int_K |u(ka_t)|^2 dk J(t) dt = \int_{B(0,R)} \mathcal{A}_2(u)(x)^2 dx. \tag{4.3.15}$$

By Lemma 4.2.1 we have $\mathcal{A}_2(u)(ka_s) \leq \mathcal{A}_{2,2}(u) e^{\alpha s}$ for all $k \in K$ and $s > 0$. This implies that for all $R > 0$ the following inequality holds,

$$(\chi_{B(0,R)} \mathcal{A}_2(u))^* \leq C \mathcal{A}_{2,2}(u) e^{\alpha R} \chi^*_{B(0,R)}.$$

Using Lemma 4.2.1, (4.3.15) and the fact $\mathcal{A}_2(u) \in L^{2,\infty}(X)$ it follows exactly as in the proof of Theorem 4.3.3 that

$$\int_{B(0,R)} |u(x)|^2 dx \leq C R \mathcal{A}_{2,2}(u)^2. \tag{4.3.16}$$

By Theorem 4.1.7 (a), there exists $f \in L^2(K/M)$ such that $u = \mathcal{P}_\lambda f$. Let us consider the case $q \neq 2$, that is, $u \in E_\lambda$ and $\mathcal{A}_q(u) \in L^{2,\infty}(X)$. Let $\{\varphi_n\} \subset C(K)$ be an approximate identity on $K$. For each fixed $t > 0$ we define the functions on $K$: $u^{a_t}(k) = u(ka_t)$, $\quad u_n^{a_t}(k) = \varphi_n * u^{a_t}(k)$, $\quad k \in K$. Since $K$ is a finite measure space, we have for $q \geq 1$, $\|u_n^{a_t}\|_{L^2(K)} \leq C \|u^{a_t}\|_{L^q(K)} \|\varphi_n\|_{L^2(K)}$. Consequently, for all $n$ we have $\mathcal{A}_2(u_n)(a_t) = \left( \int_K |u_n(ka_t)|^q dk \right)^{1/q} \leq \|\varphi_n\|_{L^2(K)} \mathcal{A}_q(u)(a_t)$. So $\mathcal{A}_2(u_n) \in L^{2,\infty}(X)$. Using the case $q = 2$ we conclude that there exists $F_n \in L^2(K/M)$ such that $u_n = \mathcal{P}_\lambda F_n$ . We also have

$$\|u_n^{a_t}\|_{L^q(K)} \leq \|\varphi_n\|_{L^1(K)} \|u^{a_t}\|_{L^q(K)} = \|u^{a_t}\|_{L^q(K)} = \mathcal{A}_q(u)(a_t), \tag{4.3.17}$$

as $\int_K \phi_n(k) dk = 1$. For $R > 0$ we define

$$g_R^n(k) = \frac{1}{R} \int_{B(0,R)} u_n(x) e^{(i\lambda-\rho)H(x^{-1}k)} dx, \; k \in K.$$

By (4.3.14) it follows that for each fixed $n$,

$$\lim_{R\to\infty} |c(\lambda)|^{-2} \int_K g_R^n(k)\overline{\psi(k)}dk = \int_K F_n(k)\overline{\psi(k)}dk, \text{ for all } \psi \in C(K). \tag{4.3.18}$$

For each $n$ we define for $\psi \in C(K)$, $L_n(\psi) = \int_K F_n(k)\overline{\psi_n(k)}dk$. We claim that $L_n \in L^{q'}(K/M)^*$. Indeed for $\psi \in C(K)$

$$\begin{aligned}
\left|\int_K g_R^n(k)\overline{\psi(k)}dk\right| &= \frac{1}{R}\left|\int_K \int_{B(0,R)} u_n(x)e^{(i\lambda-\rho)H(x^{-1}k)}\overline{\psi(k)}dxdk\right| \\
&= \frac{1}{R}\left|\int_{B(0,R)} u_n(x)\overline{\mathcal{P}_\lambda\psi(x)}dx\right| \\
&\leq \frac{1}{R}\int_0^R \left(\int_K |u_n(ka_t)|^q dk\right)^{1/q}\left(\int_K |\mathcal{P}_\lambda\psi(ka_t)|^{q'}\right)^{1/q'} J(t)dt \\
&\leq \frac{C}{R}\|\psi\|_{L^{q'}(K/M)}\int_0^R \left(\int_K |u_n(ka_t)|^q dk\right)^{1/q} e^{-\rho t}J(t)dt \text{ (by (4.3.13))} \\
&\leq \frac{C}{R}\|\psi\|_{L^{q'}(K/M)}\int_0^R \mathcal{A}_q(u)(a_t)e^{-\rho t}J(t)dt \text{ (by (4.3.17))} \\
&\leq \frac{C}{R}\|\psi\|_{L^{q'}(K/M)}\left(\int_0^R \mathcal{A}_q(u)(a_t)^2 J(t)dt\right)^{1/2}\left(\int_0^R e^{-2\rho t}J(t)dt\right)^{1/2} \\
&\leq \frac{C}{R}\|\psi\|_{L^{q'}(K/M)}\mathcal{A}_{2,q}(u)R^{1/2}R^{1/2} \text{ (by (4.3.15) and (4.3.16))} \\
&= C\|\psi\|_{L^{q'}(K/M)}\mathcal{A}_{2,q}(u)
\end{aligned}$$

From above, using (4.3.18) we conclude that $\left|\int_K F_n(k)\overline{\psi(k)}dk\right| \leq C\|\psi\|_{L^{q'}(K/M)}\mathcal{A}_{2,q}(u)$, that is, $L_n \in L^{q'}(K/M)^*$ with its norm as linear functional dominated by $C\mathcal{A}_{2,q}(u)$ for all $n$. By Eberlein-Šmulian theorem for reflexive spaces and Riesz representation theorem it follows that there exists a sequence $n_j$ and $F \in L^q(K/M)$ such that $\|F\|_{L^q(K/M)} \leq C\|\mathcal{A}_q(u)\|_{L^{2,\infty}(X)}$ and

$$\lim_{j\to\infty}\int_K F_{n_j}(k)\overline{\psi(k)}dk = \int_K F(k)\overline{\psi(k)}dk, \text{ for all } \psi \in L^{q'}(K/M).$$

Since for each fixed $x \in X$ the function $k \mapsto e^{(i\lambda-\rho)H(x^{-1}k)}$ is in $L^{q'}(K/M)$ it follows that,

$$\mathcal{P}_\lambda F(x) = \lim_{j\to\infty}\int_K F_{n_j}(k)e^{-(i\lambda+\rho)H(x^{-1}k)}dk = \lim_{j\to\infty} u_{n_j}(x) = \lim_{j\to\infty}\varphi_{n_j} * u^{a_t}(k_1) = u(k_1a_t),$$

when $x = k_1a_t$. This completes the proof. □

## 5. Roe's theorem for tempered distributions on DR spaces

This section is one of the two technical hearts of the paper. Theorems A and B and their various analogues will use the two theorems stated and proved in this section. It was mentioned in the introduction that the DR spaces have eigenfunctions (such as $x \mapsto \wp_\lambda(x,n)$ on $S$ or $x \mapsto e^{(i\lambda+\rho)H(x^{-1}k)}$ on $X$) which are not in any Lebesgue or Lorentz spaces. However they are $L^p$-tempered distributions when $\lambda \in S_p$ (see Lemma 6.1.1 (b) below) and hence theorems of this section can accommodate them. (See also subsection 6.4.)

We recall that $R$ denotes the radialization operator on the DR space $S$ (see section 2). Let $T$ be a $L^p$-tempered distribution for a fixed $p \in (1, 2]$. The distribution $T$ is called radial if

$$\langle T, \psi \rangle = \langle T, R(\psi) \rangle, \text{ for all } \psi \in C^p(S).$$

In general the radial part $R(T)$ of a $L^p$-tempered distribution $T$ is a $L^p$-tempered distribution defined by

$$\langle R(T), \psi \rangle = \langle T, R(\psi) \rangle, \text{ for all } \psi \in C^p(S).$$

Thus when $T$ itself is radial then $T = R(T)$. We shall say that $T$ has *no radial part* if $R(T) = 0$, in other words, if $\langle T, \psi \rangle = 0$ for all $\psi \in C^p(S)^\#$.

Left translation $\ell_x$ of $T$ by an element $x \in S$ is defined by the following: for $\psi \in C^p(S)$, let $\psi^*(x) = \psi(x^{-1})$. Then

$$\langle \ell_x T, \psi \rangle = T(\ell_{x^{-1}} \psi) = T * \psi^*(x^{-1}).$$

If $\psi$ is radial then $\psi^*(x) = \psi(x)$ and hence $\langle \ell_x T, \psi \rangle = T * \psi(x^{-1})$. For a radial $L^p$-tempered distribution $T$, its spherical Fourier transform $\widehat{T}$ is defined as a linear functional on $C^p(\widehat{S})^\#$ by the following rule:

$$\langle \widehat{T}, \phi \rangle = \langle T, \phi^\vee \rangle, \text{ where } \phi \in C^p(\widehat{S})^\#, \phi^\vee \in C^p(S)^\# \text{ and } \widehat{\phi^\vee} = \phi.$$

## 5.1. **Result for $L^2$-tempered distributions.**

**Theorem 5.1.1.** *Let $\{T_k\}_{k \in \mathbb{Z}}$ be a doubly infinite sequence of $L^2$-tempered distributions on $S$ satisfying*

(i) $\Delta T_k = z T_{k+1}$ *for some nonzero $z \in \mathbb{C}$ and*

(ii) *for all $\psi \in C^2(S)$, $|\langle T_k, \psi \rangle| \leq M \gamma(\psi)$ for some fixed seminorm $\gamma$ of $C^2(S)$ and $M > 0$.*

*Then*

(a) $|z| \geq \rho^2$ *implies that* $\Delta T_0 = -|z| T_0$ *and*

(b) $|z| < \rho^2$ *implies* $T_k = 0$ *for all* $k \in \mathbb{Z}$.

*Proof.* We shall divide the proof of (a) in two parts. In the first part we shall prove the assertion with the extra assumption that the distributions $T_k$ are radial.

*Case 1: $T_k$ are radial.* The argument in this part is close to one used in [20]. We shall further divide the proof for this case in a few steps.

**Step 1.** We write $z = (\alpha^2 + \rho^2) e^{i\theta}$ where $|z| = \alpha^2 + \rho^2$ for some $\alpha \geq 0$ and $\theta = \arg z$. In this step, we shall show that the distributional support of $\widehat{T_0}$ is $\{\alpha, -\alpha\}$.

It follows from hypothesis (i) of the theorem that $\Delta^k T_0 = e^{ik\theta} (\alpha^2 + \rho^2)^k T_k$ for all $k \in \mathbb{Z}$. This implies

$$\widehat{T_0} = (-1)^k e^{ik\theta} \left( \frac{\alpha^2 + \rho^2}{\lambda^2 + \rho^2} \right)^k \widehat{T_k},$$

where $\lambda$ is a dummy variable. Let $\phi \in C^2(\widehat{S})^\#$ be such that $\phi(\lambda) = 0$ if $0 \leq \lambda < \alpha + \varepsilon$. From the description of $C^2(\widehat{S})^\#$ this implies that $\phi(\lambda) = 0$ if $|\lambda| < \alpha + \varepsilon$. We claim that $\langle \widehat{T_0}, \phi \rangle = 0$.

Let $\psi \in C^2(S)^\#$ be the pre-image of $\phi(\lambda)(\alpha^2 + \rho^2)^k / (\lambda^2 + \rho^2)^k$. Then from hypothesis (i) and (ii) we get,

$$|\langle \widehat{T_0}, \phi \rangle| = |\langle \widehat{T_k}, e^{ik\theta} \left( \frac{\alpha^2 + \rho^2}{\lambda^2 + \rho^2} \right)^k \phi \rangle| = |\langle T_k, \psi \rangle| \leq M \gamma(\psi) \leq M \mu_{\beta,\tau} \left[ \left( \frac{\alpha^2 + \rho^2}{\lambda^2 + \rho^2} \right)^k \phi \right],$$

where

$$\mu_{\beta,\tau} \left[ \left( \frac{\alpha^2 + \rho^2}{\lambda^2 + \rho^2} \right)^k \phi \right] = \sup_{|\lambda| > \alpha + \varepsilon} (1 + |\lambda|)^\beta \left| \frac{d^\tau}{d\lambda^\tau} \left( \frac{\alpha^2 + \rho^2}{\lambda^2 + \rho^2} \right)^k \phi(\lambda) \right|,$$

for some positive integers $\beta$ and $\tau$. It is now easy to verify that as $k \to +\infty$,

$$\sup_{|\lambda|>\alpha+\varepsilon}(1+|\lambda|)^{\beta}|\frac{d^{\tau}}{d\lambda^{\tau}}\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k\phi(\lambda)| \to 0.$$

Again a similar argument taking $k \to -\infty$ will show that $\langle\widehat{T_0},\phi\rangle = 0$ if $\phi(\lambda)=0$ for $|\lambda|>\alpha-\varepsilon$. This establishes the claim.

**Step 2.** In this step we shall show that

$$(\Delta+\alpha^2+\rho^2)^{N+1}T_0 = 0, \text{ equivalently, } (\alpha^2-\lambda^2)^{N+1}\widehat{T_0} = 0, \text{ for some } N \in \mathbb{Z}^+. \tag{5.1.1}$$

Let $g$ be an even function in $C_c^\infty(\mathbb{R})$ such that $g \equiv 1$ on $[-1/2,1/2]$ and support of $g$ is contained in $(-1,1)$. Let $g_r$ be defined by $g_r(\xi) = g(\xi/r)$. For a function $\phi \in C^2(\widehat{S})^{\#}$ we define

$$H_r(\lambda) = (\alpha^2-\lambda^2)^{N+1}g_r(\alpha^2-\lambda^2)\phi(\lambda).$$

It is clear that $H_r \in C^2(\widehat{S})^{\#}$. Let $h_r \in C^2(S)^{\#}$ be the pre-image of $H_r$.

We fix $\varepsilon>0$ and $r(\varepsilon) = 3\varepsilon(2\alpha+\varepsilon)$. It is clear that for $\lambda \in (\alpha-\varepsilon,\alpha+\varepsilon)$ we have $g_{r(\varepsilon)}(\alpha^2-\lambda^2) \equiv 1$. Using the fact that the support of $\widehat{T_0}$ is $\{-\alpha,\alpha\}$ and $\widehat{T_0}$ acts only on even functions, we get

$$\begin{aligned} |\langle(\alpha^2-\lambda^2)^{N+1}\widehat{T_0},\phi\rangle| = |\langle\widehat{T_0},(\alpha^2-\lambda^2)^{N+1}\phi\rangle| &= |\langle\widehat{T_0},H_{r(\varepsilon)}\rangle| = |\langle T_0,h_{r(\varepsilon)}\rangle| \\ &\leq M\gamma(h_{r(\varepsilon)}) \leq M\mu_{\beta,\tau}(H_{r(\varepsilon)}), \end{aligned}$$

for some positive integers $\beta,\tau$. The proof of this step will be completed if we show that

$$\sup_{\lambda\in\mathbb{R}}(1+|\lambda|)^{\beta}\left|\frac{d^{\tau}}{d\lambda^{\tau}}((\alpha^2-\lambda^2)^{N+1}g_{r(\varepsilon)}(\alpha^2-\lambda^2)\phi(\lambda)\right| \tag{5.1.2}$$

converges to zero as $\varepsilon \to 0$ for a suitably large $N$.

We note that $|d^s/d\lambda^s\phi(\lambda)| \leq M$ for some $M>0$ for all $s\leq\tau$, $|d^s/d\lambda^s g_{r(\varepsilon)}(\lambda)| \leq B/|\lambda|^s$ for some $B>0$ for all $s\leq\tau$ and the function $G_{r(\varepsilon)}(\lambda) = g_{r(\varepsilon)}(\alpha^2-\lambda^2)$ along with its derivatives vanishes when $|\alpha^2-\lambda^2| > r(\varepsilon)$.

Therefore in the expression (5.1.2) the supremum can be taken over $\{\lambda\in\mathbb{R} \mid |\alpha^2-\lambda^2| < r(\varepsilon)\}$ and thus it is dominated by $C_\phi|\alpha^2-\rho^2|^\beta \leq C_\phi r(\varepsilon)^\beta$ for some $\beta>0$ and hence converges to zero as $\varepsilon\to 0$.

This shows that $|\langle(\alpha^2-\rho^2)^{N+1}\widehat{T_0},\phi\rangle| = 0$ which completes Step 2.

**Step 3:** We shall establish that

$$(\Delta+\alpha^2+\rho^2)T_0 = 0.$$

From (5.1.1) it follows that

$$\mathrm{Span}\{T_0,T_1,\cdots\} = \mathrm{Span}\{T_0,\Delta T_0,\cdots,\Delta^N T_0\} = \mathrm{Span}\{T_0,T_1,\cdots,T_N\}.$$

Suppose that $(\Delta+\alpha^2+\rho^2)T_0 \neq 0$. Let $k_0$ be the largest positive integer such that $(\Delta+\alpha^2+\rho^2)^{k_0}T_0 \neq 0$. Then $k_0 \leq N$.

Let $g = (\Delta+\alpha^2+\rho^2)^{k_0-1}T_0 \in$ Span $\{T_0,T_1,\cdots,T_N\}$. We assume that $g = a_0T_0+\cdots+a_NT_N$. Then

$$(\Delta+\alpha^2+\rho^2)^2g = (\Delta+\alpha^2+\rho^2)^{k_0+1}T_0 = 0, (\Delta+\alpha^2+\rho^2)g = (\Delta+\alpha^2+\rho^2)^{k_0}T_0 \neq 0. \tag{5.1.3}$$

Using binomial expansion and (5.1.3) we get for any positive integer $k$,

$$\begin{aligned}\Delta^k g &= (\Delta + (\alpha^2+\rho^2) - (\alpha^2+\rho^2))^k g \\ &= k(-1)^{k-1}(\alpha^2+\rho^2)^{k-1}(\Delta + (\alpha^2+\rho^2))g + (-1)^k(\alpha^2+\rho^2)^k g.\end{aligned}$$

This implies for any $\psi \in C^2(S)^\#$,

$$|\langle(\Delta + (\alpha^2+\rho^2))g, \psi\rangle| \leq \frac{1}{k}(\alpha^2+\rho^2)^{1-k}|\langle\Delta^k g, \psi\rangle| + \frac{1}{k}(\alpha^2+\rho^2)|\langle g, \psi\rangle|. \tag{5.1.4}$$

Since,

$$\begin{aligned}|\langle\Delta^k g, \psi\rangle| &= |\langle\Delta^k(a_0T_0 + a_1T_1 + \cdots + a_NT_N), \psi\rangle| \\ &= |a_0(\alpha^2+\rho^2)^k e^{ik\theta}\langle T_k, \psi\rangle + \cdots + a_N(\alpha^2+\rho^2)^k e^{ik\theta}\langle T_{N+k}, \psi\rangle| \\ &\leq (\alpha^2+\rho^2)^k|a_0\langle T_k, \psi\rangle| + \cdots + |a_N\langle T_{N+k}, \psi\rangle| \\ &\leq M(\alpha^2+\rho^2)^k(|a_0| + \cdots + |a_N|)\gamma(\psi),\end{aligned}$$

from above and by (5.1.4) it follows that,

$$|\langle(\Delta + (\alpha^2+\rho^2))g, \psi\rangle| \leq M\frac{(\alpha^2+\rho^2)}{k}(|a_0| + \cdots + |a_N|)\gamma(\psi) + \frac{(\alpha^2+\rho^2)}{k}|\langle g, \psi\rangle|$$

and the right side goes to 0 as $k \to \infty$. Therefore by (5.1.3) $(\Delta + (\alpha^2+\rho^2))^{k_0}T_0 = 0$ which contradicts the assumption on $k_0$. Thus we have shown that $N = 0$, i.e., $(\Delta + \alpha^2 + \rho^2)T_0 = 0$. This completes the proof of (a) for radial $T_k$.

Now we shall withdraw the assumption of radiality from the sequence $\{T_k\}$.

*Case 2: $T_k$ are not necessarily radial.* In this proof we shall frequently use the fact that $\Delta$ commutes with the radialization and with translations (see section 2). The following steps will lead to the proof.

**Step** $1'$. Given any $k \in \mathbb{Z}$, there is a $x \in S$ such that $\ell_x T_k$ has nonzero radial part. Indeed if $R(\ell_x T_k) = 0$ for all $x \in S$, then $\langle \ell_x T_k, h_t\rangle = 0$ for all $t > 0$ where $h_t$ denotes the heat kernel, which is a radial function (see [3]). That is $T_k * h_t \equiv 0$. But $T_k * h_t \to T_k$ as $t \to 0$ in the sense of distribution. Therefore $T_k = 0$ and there is nothing to prove. We note that this also shows that if for two $L^2$-tempered distribution $T$ and $T'$, $R(\ell_x T) = R(\ell_x T')$ for all $x \in S$, then $T = T'$.

**Step** $2'$. We claim that if $R(\ell_y T_0) \neq 0$ for some $y \in S$, then $R(\ell_y T_k) \neq 0$ for all $k \in \mathbb{Z}$. It is enough to show that if $R(\ell_y T_0) \neq 0$ then $R(\ell_y T_{-1}) \neq 0$ and $R(\ell_y T_1) \neq 0$. Indeed if $R(\ell_y T_{-1}) = 0$ then $\Delta R(\ell_y T_{-1}) = 0$ which implies $R(\ell_y T_0) = 0$ as $\Delta T_{-1} = zT_0$ for $z \neq 0$.

If $R(\ell_y T_1) = 0$, then $\langle \ell_y T_1, \psi\rangle = 0$ for all $\psi \in C^2(S)^\#$. That is $\langle \ell_y \Delta T_0, \psi\rangle = 0$ and hence $\langle \ell_y T_0, \Delta\psi\rangle = 0$. Since for any $\phi \in C^2(S)^\#$, $\widehat{\phi}(\lambda)(\lambda^2+\rho^2)^{-1} \in C^2(\widehat{S})^\#$ (see section 2), $\phi$ can be written as $\phi = \Delta\psi$ for some $\psi \in C^2(S)^\#$. Thus $\langle \ell_y T_0, \phi\rangle = 0$ for any $\phi \in C^2(S)^\#$, i.e. $R(\ell_y T_0) = 0$.

**Step** $3'$. In this step we shall show that for any $y \in S$, the sequence $\{R\ell_y T_k\}$ of radial distributions satisfies the hypothesis of the theorem. Since $\Delta$ commutes with radialization and translations it follows from the hypothesis $\Delta T_k = zT_{k+1}$ that $\Delta R(\ell_y T_k) = zR(\ell_y T_{k+1})$.

It now remains to show that for the seminorm $\gamma$ of $C^2(S)$ in the hypothesis of the theorem and $\psi_1 \in C^2(S)^\#$, $|\langle R(\ell_y T_k), \psi_1\rangle| \leq C_y M\gamma(\psi_1)$. Using the estimate $\phi_0(x)^{-1}(1+|x|) \asymp e^{\rho|x|}$ we have for

any seminorm $\gamma$ of $C^2(S)$ and $\psi \in C^2(S)$,

$$\begin{aligned}
\gamma(\ell_y\psi) &= \sup_{x\in S} |D\psi(y^{-1}x)|\phi_0(x)^{-1}(1+|x|)^L \\
&= \sup_{x\in S} |D\psi(x)|\phi_0(yx)^{-1}(1+|yx|)^L \\
&\asymp \sup_{x\in S} |D\psi(x)|e^{\rho|yx|}(1+|yx|)^{L-1} \\
&\leq e^{\rho|y|}(1+|y|)^{L-1} \sup_{x\in S} |D\psi(x)|e^{\rho|x|}(1+|x|)^{L-1} \\
&\asymp e^{\rho|y|}(1+|y|)^{L-1} \sup_{x\in S} |D\psi(x)|\phi_0(x)^{-1}(1+|x|)^L \\
&= C_y\gamma(\psi),
\end{aligned}$$

where the constant $C_y$ depends only on $y \in S$. Since $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ for any $\psi \in C^2(S)$, it follows that for $\psi_1 \in C^2(S)^\#$,

$$|\langle R(\ell_y T_k), \psi_1\rangle| = |\langle \ell_y T_k, \psi_1\rangle| = |\langle T_k, \ell_{y^{-1}}\psi_1\rangle| \leq M\gamma(\ell_{y^{-1}}\psi_1) \leq C_y M\gamma(\psi_1).$$

**Step** $4'$. From the previous step and the result proved for radial distributions we conclude that

$$\Delta R(\ell_y(T_0)) = -|z|R(\ell_y(T_0)) \text{ for all } y \in S.$$

(Note that if $R(\ell_y(T_0)) = 0$ for some $y \in S$, then the identity $\Delta R(\ell_y(T_0)) = -|z|R(\ell_y(T_0))$ is trivial.) Again appealing to the fact that $\Delta$ commutes with translations and radialization we have $R(\ell_y(\Delta T_0)) = R(\ell_y(-|z|T_0))$ for all $y \in S$. By Step $1'$ this implies $\Delta T_0 = -|z|T_0$ which is the assertion of (a).

To prove part (b) of the theorem we note again that for an $L^2$-tempered distribution $T$, $T = 0$ is equivalent to $R(\ell_x T) = 0$ for all $x \in S$ and that $\Delta T = \alpha T$ implies $\Delta R(\ell_x T) = \alpha R(\ell_x T)$ for all $x \in S$. Therefore it is enough to assume that $T_k$ are radial. We can proceed as in the proof of (a) (for radial distributions). For $\phi \in C^2(\widehat{S})^\#$ and suitable seminorms $\gamma$ and $\mu$ of $C^2(S)$ and $C^2(\widehat{S})^\#$ respectively,

$$\begin{aligned}
|\langle \widehat{T_0}, \phi\rangle| = \left|\left\langle \widehat{T_k}, \frac{z^k}{(\lambda^2+\rho^2)^k}\phi\right\rangle\right| = \left|\left\langle T_k, \left(\frac{z^k}{(\lambda^2+\rho^2)^k}\phi\right)^\vee\right\rangle\right| &\leq M\gamma\left[\left(\frac{z^k}{(\lambda^2+\rho^2)^k}\phi\right)^\vee\right] \\
&\leq M\mu\left[\frac{z^k}{(\lambda^2+\rho^2)^k}\phi\right].
\end{aligned}$$

Since $|z| < |\lambda^2+\rho^2|$ for all $\lambda \in S_2 = \mathbb{R}$, the right side goes to 0 as $k \to \infty$ and we conclude that $\langle T_0, \psi\rangle = 0$ for all $\psi \in C^2(S)^\#$. □

**Remark 5.1.2.** For the symmetric spaces $X$, since $\Delta$ preserves the $K$-types and since $L^p$-Schwartz space isomorphism theorems are available for functions on $X$ with a single left $K$-type (see e.g. [24]), we can decompose the distribution $T_k$ in $K$-isotypic components and work restricting to one isotypic component at a time exactly the same way we worked with the radial functions in the previous theorem. This is an alternative way to prove Theorem 5.1.1. However, this method has no relevance for a general $NA$ group, where the rotation group $K$ is not available.

5.2. **Result for $L^p$-tempered distributions for** $1 < p < 2$**.** A distinguishing feature of the corresponding theorem for the $L^p$-tempered distributions is that here a one-sided sequence of functions will be enough. Handling this case becomes technically more challenging because the $L^p$-spectrum of $\Delta$ is a parabolic region in the complex plane. Thus one has to enter into the realm of holomorphic functions and the main technique in the proof of the previous theorem, namely the use of functions whose Fourier transforms are "supported outside a set of positive measure" will not work.

**Theorem 5.2.1.** *For $1 < p < 2$, let $\{T_k\}_{k\in\mathbb{Z}^+}$ be an infinite sequence of $L^p$-tempered distributions on $S$ satisfying*

(i) $\Delta T_k = zT_{k+1}$ *for some nonzero* $z \in \mathbb{C}$ *and*

(ii) *for all* $\psi \in C^p(S)$, *let* $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ *for some fixed seminorm* $\gamma$ *of* $C^p(S)$ *and* $M > 0$. *Then*

(a) $|z| = 4\rho^2/pp'$ *implies that* $\Delta T_0 = -|z|T_0$,

(b) $|z| < 4\rho^2/pp'$ *implies that* $T_k = 0$ *for all* $k \in \mathbb{Z}$ *and*

(c) *there are solution which are not eigendistributions when* $|z| > 4\rho^2/pp'$.

*Proof.* (a) It suffices to prove the theorem with the extra assumption that $T_k$ are radial as it is possible to extend the theorem from the radial to the general case using the argument given in the proof of Theorem 5.1.1

For notational convenience in this proof let us write $\alpha$ for $i\gamma_p\rho$. Then $|z| = \alpha^2 + \rho^2$. We shall show that $(\alpha^2 - \lambda^2)^{N+1}\widehat{T_0} = 0$ for a fixed $N \in \mathbb{Z}^+$. Since $\Delta^k T_0 = z^k T_k$ and for any $\lambda \in S_p$, $\lambda^2 + \rho^2 \neq 0$, we have $\widehat{T_0} = z^k(-1)^k(\lambda^2 + \rho^2)^{-k}\widehat{T_k}$ and hence for a fixed $\phi \in C^p(\widehat{S})^\#$,

$$\begin{aligned}
|\langle(\alpha^2 - \lambda^2)^{N+1}\widehat{T_0}, \phi\rangle| &= |\langle\widehat{T_0}, (\alpha^2 - \lambda^2)^{N+1}\phi\rangle| \\
&= |\langle\widehat{T_k}, \left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k (\alpha^2 - \lambda^2)^{N+1}\phi\rangle| \\
&= \left|\left\langle T_k, \left(\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k (\alpha^2 - \lambda^2)^{N+1}\phi\right)^\vee\right\rangle\right| \\
&\leq M\gamma\left[\left(\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k (\alpha^2 - \lambda^2)^{N+1}\phi\right)^\vee\right] \\
&\leq M\,\mu\left[\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k (\alpha^2 - \lambda^2)^{N+1}\phi\right],
\end{aligned}$$

where $\mu$ is given by

$$\mu\left[\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k (\alpha^2 - \lambda^2)^{N+1}\phi(\lambda)\right] = \sup_{\lambda\in S_p}\left|\frac{d^\tau}{d\lambda^\tau}P(\lambda)\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k (\alpha^2 - \lambda^2)^{N+1}\phi(\lambda)\right|,$$

for some even polynomial $P(\lambda)$ and derivative of even order $\tau$. We shall temporarily use the notation

$$F^k = \left|\frac{d^\tau}{d\lambda^\tau}P(\lambda)\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k (\alpha^2 - \lambda^2)^{N+1}\phi\right|.$$

We shall use the notation $A_\tau, B_\tau, C_\tau, C'_\tau$ etc. for positive constants which depend only on $\tau$.

Our aim is to show that $\sup_{\lambda\in S_p} F^k \to 0$ as $k \to \infty$. We note that in the definition of $\mu$ we can take the supremum only on $S_p^+ = \{\lambda \in S_p \mid \Im\lambda \geq 0\}$ as $\phi$ is even being the image of a radial function. We also need the following observations: For $\lambda \in S_p^+$,

$$\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right| \leq 1, \; \left|\frac{\alpha^2-\lambda^2}{\lambda^2+\rho^2}\right| = \left|\frac{\alpha^2+\rho^2-(\lambda^2+\rho^2)}{\lambda^2+\rho^2}\right| < 1. \tag{5.2.1}$$

For $\lambda \in S_p^+$ we write $\lambda = a\rho + ib\rho$. Then $a \in \mathbb{R}$ and $0 \leq b \leq \gamma_p$ and hence,

$$\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right| \leq \frac{1-\gamma_p^2}{1+a^2-b^2}. \tag{5.2.2}$$

We fix $N = 6\tau + 1$. Let

$$A_\tau = \max_{i=1}^{\tau}\{\sup_{\lambda\in S_p^+} |\frac{d^i}{d\lambda^i}(\alpha^2-\lambda^2)^{N+1}P(\lambda)\phi(\lambda)| \mid 0\leq i\leq \tau\}.$$

An explicit computation shows that for $\lambda \in S_p^+$.

$$|\frac{d^i}{d\lambda^i}\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^k| \leq B_\tau \left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right|^k k(k+1)\dots(k+i-1), \quad 0\leq i\leq\tau. \tag{5.2.3}$$

Therefore

$$\left|\frac{d^\tau}{d\lambda^\tau}\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^{k+N+1}(\alpha^2-\lambda^2)^{N+1}P(\lambda)\phi(\lambda)\right|$$

$$\leq \sum_{i=0}^{\tau}\binom{\tau}{i}B_\tau\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right|^{k+N+1}(k+N+1)\dots(k+N+i)\left|\frac{d^{\tau-i}}{d\lambda^{\tau-i}}(\alpha^2-\lambda^2)^{N+1}P(\lambda)\phi(\lambda)\right| \tag{5.2.4}$$

$$\leq A_\tau C_\tau k^\tau\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right|^k\left|\frac{\alpha^2-\lambda^2}{\lambda^2+\rho^2}\right|^{N+1}$$

$$\leq A_\tau C_\tau k^\tau\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right|^k. \tag{5.2.5}$$

For any $k = 1, 2, \dots$ we define a bounded region,

$$V_k = \{z\in S_p^+ \mid |\Re z| < k^{-1/4} \text{ and } \gamma_p\rho - \Im z < k^{-1/4}\}.$$

Our strategy is to show that as $k\to\infty$,
(i) $\sup_{\lambda\in V_k} F^{k+N+1}\to 0$ (ii) $\sup_{\lambda\in V_k^c} F^{k+N+1}\to 0$ uniformly on $k$.
We shall first deal with (ii) above.
We claim that for all $\lambda\in V_k^c$ and large $k$,

$$\left|\frac{d^\tau}{d\lambda^\tau}\left(\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right)^{k+N+1}(\alpha^2-\lambda^2)^{N+1}P(\lambda)\phi(\lambda)\right| \leq C'_\tau\left(1+\frac{c}{\sqrt{k}}\right)^{-k}k^\tau, \tag{5.2.6}$$

for some constant $c > 0$.

In view of (5.2.5) it suffices to show that for some constant $c>0$,

$$\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right|^k \leq \left(1+\frac{c}{\sqrt{k}}\right)^{-k}.$$

Since $\lambda\in V_k^c$, there are two possibilities: $|a|\geq k^{-1/4}/\rho$ or $\gamma_p - b\geq k^{-1/4}$.
**Case 1:** $|a|\geq k^{-1/4}/\rho$. Using $-b^2 > -\gamma_p^2$ and (5.2.2), we get,

$$\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right| \leq \frac{1-\gamma_p^2}{1+a^2-b^2} \leq \frac{1-\gamma_p^2}{1-\gamma_p^2+a^2} = \left(1+\frac{a^2}{1-\gamma_p^2}\right)^{-1} \leq \left(1+\frac{k^{-1/2}/\rho^2}{1-\gamma_p^2}\right)^{-1} = \left(1+\frac{c_1}{\sqrt{k}}\right)^{-1},$$

where $c_1 = (\rho^2-\gamma_p^2\rho^2)^{-1}$.
**Case 2:** $\gamma_p - b\geq k^{-1/4}$. Then $\gamma_p^2 - b^2 = (\gamma_p+b)(\gamma_p-b)\geq \gamma_p k^{-1/4}$. Hence

$$\frac{1-\gamma_p^2}{1+a^2-b^2} \leq \frac{1-\gamma_p^2}{1-b^2} = \frac{1-\gamma_p^2}{1-\gamma_p^2+(\gamma_p^2-b^2)} \leq \left(1+\frac{c_2}{k^{1/4}}\right)^{-1},$$

where $c_2 = \gamma_p/(1-\gamma_p^2)$.

Choosing $c = \min\{c_1, c_2\}$ from the two cases above, we get that for all $\lambda\in V_k^c$,

$$\left|\frac{\alpha^2+\rho^2}{\lambda^2+\rho^2}\right|^k \leq \left(1+\frac{c}{\sqrt{k}}\right)^{-k}.$$

Thus (5.2.6) is proved from which the assertion (ii) follows easily.

To prove (i) we take $\lambda \in V_k$. Then $|\Re(\alpha - \lambda)| = |\Re\lambda| < k^{-1/4}$ and $|\Im(\alpha - \lambda)| = |\gamma_p\rho - b\rho| < k^{-1/4}$ and hence $|\alpha - \lambda| < \sqrt{2}k^{-1/4}$. Using $|a\rho| < k^{-1/4}$ and $0 \leq b\rho < \gamma_p\rho$ we get,

$$|\alpha + \lambda|^2 = |a\rho + i(b\rho + \gamma_p\rho)|^2 = a^2\rho^2 + b^2\rho^2 + \gamma_p^2\rho^2 + 2b\rho\gamma_p\rho < 1/\sqrt{k} + 4\gamma_p^2\rho^2.$$

Therefore for $\lambda \in V_k$, $|\alpha^2 - \lambda^2| = |\alpha - \lambda||\alpha + \lambda| < (1/\sqrt{k} + 4\gamma_p^2\rho^2)^{1/2}\sqrt{2}k^{-1/4}$ and hence

$$|\alpha^2 - \lambda^2|^{4\tau}(k + N + 1)(k + N + 2)\dots(k + N + \tau) \leq C_\tau.$$

From (5.2.4) we see that every term in $F^{k+N+1}$ contains the factor $(\alpha^2 - \lambda^2)^{N+1-\tau} = (\alpha^2 - \lambda^2)^{5\tau+2}$. The inequalities (5.2.1), (5.2.4) and $|\alpha^2 - \lambda^2|^{\tau+1} \leq Ck^{-(\tau+1)/4}$, now implies that $F_{k+N+1} \leq Ck^{-(\tau+1)/4}$ where $C$ is independent of $k$. Thus assertion (i) is proved. It follows from (i) and (ii) that $\sup_{\lambda \in S_p} F^k \to 0$ as $k \to \infty$ and hence $(\alpha^2 - \lambda^2)^{N+1}\widehat{T_0} \equiv 0$, when $T_k$ are radial. Now the argument given in Step 3 of the previous theorem will lead to the theorem restricted to radial distributions and hence for the general case.

The proof of part (b) is also similar to that of part (b) of Theorem 5.1.1. By the argument given there it is enough to consider that $T_k$ are radial. If we proceed through the steps of the proof of (a) we get, for $\phi \in C^p(\widehat{S})^\#$, $|\langle\widehat{T_0}, \phi\rangle| \leq \mu[z^k(\lambda^2 + \rho^2)^{-k}\phi]$, for some seminorm $\mu$ of $C^p(\widehat{S})^\#$. Since $|z| < 4\rho^2/pp' \leq |\lambda^2 + \rho^2|$ for all $\lambda \in S_p$, the right side goes to zero as $k \to \infty$. Hence $\langle T_0, \psi\rangle = 0$ for all $\psi \in C^p(S)^\#$.

(c) If $|z| > 4\rho^2/pp'$ then one can find $\theta_1$ and $\theta_2$ such that $|z|e^{i\theta_1}$ and $|z|e^{i\theta_2}$ are in the interior of the $L^p$-spectrum. The conclusion now follows by arguing as in subsection 3.2. □

## 6. Proof of the Theorems A, B and their analogues

6.1. **Functions which are tempered distributions.** It follows easily from the definition of the $L^1$-Schwartz space $C^1(S)$ that $C^1(S) \subset L^1(S)$ and hence all $L^\infty$-functions on $S$ define $L^1$-tempered distribution, i.e., $L^\infty(S) \subset C^1(S)'$. We shall see that the size estimates $L^{p',\infty}$, $M_{p'}$ and $\mathcal{A}_{p',q}$ define $L^p$-tempered distributions for $1 < p \leq 2$.

**Lemma 6.1.1.** *Let $f$ be a measurable function on $S$ and $1 < p \leq 2, 1 < q < \infty$.*
(a) *There exists a seminorm $\gamma$ of $C^p(S)$ such that for all $\phi \in C^p(S)$ and suitable functions $f$,*

$$\text{(i)}\quad |\langle f, \phi\rangle| \leq C\|f\|_{p',\infty}\gamma(\phi),\ \text{(ii)}\ |\langle f, \phi\rangle| \leq CM_{p'}(f)\gamma(\phi),\ \text{(iii)}\ |\langle f, \phi\rangle| \leq C\mathcal{A}_{p',q}(f)\gamma(\phi).$$

(b) *If $\lambda \in S_p$ then for each $n \in N$, $x \mapsto \wp_\lambda(x, n)$ and for each $k \in K$, $x \mapsto e^{(i\lambda+\rho)H(x^{-1}k)}$ are $L^p$-tempered distributions.*

*Proof.* (a) We fix $p \in (1, 2]$ and $L > \max\{\rho, 1 + 1/2p\}$. We define a seminorm $\gamma$ on $C^p(S)$ by $\gamma(\phi) = \sup_{x\in S} |\phi(x)|\phi_0(x)^{-2/p}(1 + |x|)^{2L}$, $\phi \in C^p(S)$. We shall first show that the radial function $h(x) = \phi_0(x)^{2/p}(1 + |x|)^{-2L}$ is in $L^{p,1}(S)$, which is equivalent to showing that $h(r) = e^{-\frac{2\rho}{p}r}(1 + r)^{-2L}$ is in $L^{p,1}([0, \infty), J(r)dr)$ where $J(r)$ is the Jacobian of the Haar measure in polar decomposition (see (2.2.2), (2.3.2)). A lengthy but routine computation shows that the decreasing rearrangement $h^*$ of $h$ satisfies

$$\text{(6.1.1)}\qquad h^*(t) \asymp 1, \quad t \in (0, 1] \ \text{ and } \ h^*(t) = h^*(e^{2\rho u}) \asymp e^{-\frac{2\rho}{p}u}(1 + u)^{-L/\rho} \quad u \geq 0.$$

Since $L > \rho$ from above we obtain

$$\begin{aligned}\int_0^\infty h^*(t)t^{-1/p'}dt &\asymp \int_0^1 t^{-1/p'}dt + \int_1^\infty h^*(t)t^{-1/p'}dt \\ &= \int_0^1 t^{-1/p'}dt + 2\rho\int_0^\infty h^*(e^{2\rho y})e^{-\frac{2\rho y}{p'}}e^{2\rho y}dy \\ &= \int_0^1 t^{-1/p'}dt + 2\rho\int_0^\infty (1+y)^{-L/\rho}dy < \infty.\end{aligned}$$

Thus $h \in L^{p,1}(S)$. From this (i) and (iii) follows easily. Indeed,

$$\left|\int_S f(x)\phi(x)(x)dx\right| \leq \int_S |\phi(x)|\phi_0(x)^{-2/p}(1+|x|)^{2L}|f(x)|h(x)dx \leq \gamma(\phi)\|f\|_{p',\infty}\|h\|_{p,1} \text{ and}$$

$$\begin{aligned}|\int_S f(x)\phi(x)dx| &\leq \gamma(\phi)\int_S |f(x)|h(x)dx \\ &= \gamma(\phi)\int_0^\infty\int_{\partial B(\mathfrak{s})}|f(r\omega)|h(r)d\omega\, J(r)dr \\ &\leq \gamma(\phi)\int_0^\infty \mathcal{A}_q(f)(a_r)h(r)J(r)dr \\ &\leq \gamma(\phi)\mathcal{A}_{p',q}(f)\|h\|_{p,1}.\end{aligned}$$

(ii) By definition of $M_{p'}(f)$ there exists a natural number $n_0$ such that $\int_{B(0,R)}|f(x)|^{p'}dx \leq 2RM_{p'}(f)^{p'}$, for all $R \geq n_0$. We fix one such $n_0$. Since $\phi_0(a_r) \asymp e^{-\rho|r|}(1+|r|)$, we have for $L$ as above

$$\begin{aligned}(6.1.2)\qquad & \int_{B(0,n_0)^c}|f(x)|\frac{\phi_0(x)^{2/p}}{(1+|x|)^{2L}}dx \\ \asymp\ & \int_{n_0}^\infty\int_{\partial B(\mathfrak{s})}|f(r\omega)|\frac{e^{\frac{-2\rho}{p}r}}{(1+r)^{2L-2/p}}e^{2\rho r}d\omega dr \\ \leq\ & \int_{n_0}^\infty\left(\int_{\partial B(\mathfrak{s})}|f(r\omega)|^{p'}d\omega\right)^{1/p'}\frac{e^{\frac{2\rho}{p'}r}}{(1+r)^{2L-2/p}}dr \\ =\ & \lim_{N\to\infty}\sum_{k=0}^{N-1-n_0}\int_{n_0+k}^{n_0+k+1}\left(\int_{\partial B(\mathfrak{s})}|f(r\omega)|^{p'}d\omega\right)^{1/p'}\frac{e^{\frac{2\rho}{p'}r}}{(1+r)^{2L-2/p}}dr \\ \leq\ & \lim_{N\to\infty}\sum_{k=0}^{N-1-n_0}\frac{1}{(1+n_0+k)^{2L-2/p}}\left(\int_{n_0+k}^{n_0+k+1}\int_{\partial B(\mathfrak{s})}|f(r\omega)|^{p'}e^{2\rho r}d\omega dr\right)^{1/p'} \\ \leq\ & 2M_{p'}(f)\lim_{N\to\infty}\sum_{k=0}^{N-1-n_0}\frac{1}{(1+n_0+k)^{2L-1-1/p}} \leq CM_{p'}(f)\end{aligned}$$

as $2L > 2+1/p$. On the other hand

$$(6.1.3)\qquad \left|\int_{B(0,n_0)}f(x)\frac{\phi_0(x)^{2/p}}{(1+|x|)^{2L}}dx\right| \leq C\left(\int_{B(0,n_0)}|f(x)|^{p'}dx\right)^{1/p'} \leq Cn_0^{1/p'}M_{p'}(f).$$

Combining (6.1.2) and (6.1.3) we have for $\phi \in C^p(S)$,

$$\left|\int_S f(x)\phi(x)dx\right| = \left|\int_S f(x)\frac{\phi_0(x)^{2/p}}{(1+|x|)^{2L}}(1+|x|)^{2L}\phi_0(x)^{-2/p}\phi(x)dx\right| \leq CM_{p'}(f)\gamma(\phi).$$

(b) We recall that for $\lambda \in S_p$, $\phi_\lambda \in L^{p',\infty}(S)$, $[R\wp_\lambda(\cdot,n)](x) = \wp_\lambda(e,n)\phi_\lambda(x)$ ([4, p. 410]) and the radial function $h(x) = \phi_0(x)^{2/p}(1+|x|)^{-2L} \in L^{p,1}(S)$ (see (a)). Hence for $\phi \in C^p(S)$ and $\lambda \in S_p$,

$$|\int_S \phi(x)\wp_\lambda(x,n)dx| \leq \gamma(\phi)|\int_S h(x)\wp_\lambda(x,n)dx| = |\wp_\lambda(e,n)|\gamma(\phi)|\int_S h(x)\phi_\lambda(x)dx| < \infty.$$

Similarly using $\int_K e^{(i\lambda+\rho)H(x^{-1}k)}dk = \phi_\lambda(x)$ we get the other assertion. □

6.2. **Completion of proofs.** Before we enter the proofs we need to explain the statements of Theorem A and B. Indeed it is clear that both $\Delta$ and $\Delta^{-1}$ act as radial $L^p$-multipliers and hence a radial $L^{p'}$-multiplier for $1 < p < 2$. (See [2, Theorem 1], [3, Corollary 4.18].) Hence by interpolation ([38, p. 197]), they act as radial $L^{p,\infty}$-multiplier for $1 < p < \infty$. We also note that Lemma 6.1.1 and the hypotheses of Theorem A and B ensure that the function $f$ is a $L^p$-tempered distribution. It follows from the definition of $C^p(S)$ that for $\phi \in C^p(S)$, both $\Delta\phi$ and $\Delta^{-1}\phi$ are functions in $C^p(S)$, where the latter is interpreted as a radial multiplier on $C^p(S)$. Hence $\Delta^k f, k \in \mathbb{Z}$ can also be considered in the sense of $L^p$-tempered distributions. i.e. $\langle \Delta^k f, \phi\rangle = \langle f, \Delta^k\phi\rangle$. Lemma 6.1.1 shows that this distributional interpretation is more robust and is applicable to the various analogues of Theorem A and B, which will be discussed in the next subsection.

*Proof of Theorem B.* Let $T_k = (4\rho^2/pp')^{-k}\Delta^k f$ for all $k \in \mathbb{Z}^+$. From the hypothesis we have $(\alpha^2 + \rho^2)^{-k}\Delta^k f \in L^{p',\infty}(S)$. Therefore by Lemma 6.1.1 (a), $T_k$ is an $L^p$-tempered distribution and $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ for all $\psi \in C^p(S)$. We also note that

$$\Delta T_k = \left(\frac{4\rho^2}{pp'}\right)^{-k}\Delta^{k+1}f = \left(\frac{4\rho^2}{pp'}\right)\left(\frac{4\rho^2}{pp'}\right)^{-(k+1)}\Delta^{k+1}f = \left(\frac{4\rho^2}{pp'}\right)T_{k+1}.$$

Thus the sequence $\{T_k\}$ satisfies the hypothesis of Theorem 5.2.1 (a) and hence $\Delta T_0 = -4\rho^2/pp'T_0$, i.e. $\Delta f = -4\rho^2/pp'f$.

If $S$ is a symmetric space then by Corollary 4.1.5, we have $f = \mathcal{P}_{i\gamma_{p'}\rho}F$ for some $F \in L^{p'}(K/M)$. □

Similarly, Theorem A follows from Lemma 6.1.1 (a), Theorem 5.1.1 and Theorem 4.3.5.

6.3. **Other Analogs.** It is easy to observe the following:

(i) Applying Lemma 6.1.1 (a) and Theorem 4.1.7 in Theorem 5.1.1, we obtain a version of Theorem A substituting $L^{2,\infty}$-norm by $M_2$ norm.
(ii) Applying Lemma 6.1.1 (a) and Theorem 4.3.6 in Theorem 5.1.1, we obtain a version of Theorem A for hyperbolic spaces substituting $L^{2,\infty}$-norm by $\mathcal{A}_{2,q}$-norm with $1 < q < \infty$.
(iii) Applying Lemma 6.1.1 (a) and Theorem 4.3.1 in Theorem 5.2.1, we get a version of Theorem B, substituting $L^{p',\infty}$-norm by $M_{p'}$-norm for $1 < p < 2$.
(iv) Applying Lemma 6.1.1 (a) and Theorem 4.3.3 in Theorem 5.2.1, a version of Theorem B is obtained where $L^{p',\infty}$-norm is substituted by $\mathcal{A}_{p',q}$-norm for $1 < p < 2$, $1 < q < \infty$.

Theorem 5.1.1 and Theorem 5.2.1 and the results of section 4 also yield the following versions of Theorem A and Theorem B. Notice that these theorems resemble Theorem 1.0.1 as $L^\infty$-norm is in use.

**Theorem 6.3.1.** *For a measurable functions $f$ on $S$ and $\lambda > 0$, if $\|\phi_\lambda^{-1}\Delta^k f\|_\infty \leq C_\lambda(\lambda^2+\rho^2)^k$ for all $k \in \mathbb{Z}$, then $\Delta f = -(\lambda^2+\rho^2)f$. When $S$ is a symmetric space then, for $\lambda \neq 0$, $f = \mathcal{P}_\lambda F$ for some $F \in L^2(K/M)$ and for $\lambda = 0$, $f = \mathcal{P}_0 F$ for some $F \in L^\infty(K/M)$.*

*Proof.* It is easy to see that for any $\phi \in C^2(S)$, $|\langle \Delta^k f, \phi\rangle| \leq \gamma(\phi)\|\phi_0^{-1}\Delta^k f\|_\infty \leq \gamma(\phi)\|\phi_\lambda^{-1}\Delta^k f\|_\infty$. Therefore if $T_k = (\lambda^2+\rho^2)^{-1}\Delta^k f$ then $T_k$ are $L^2$-tempered distributions and satisfies the hypothesis of Theorem 5.1.1. Therefore we obtain, $\Delta f = -(\lambda^2+\rho^2)f$. If $\lambda \neq 0$, then $\phi_\lambda \in L^{2,\infty}(S)$ and hence $f \in L^{2,\infty}(S)$. Therefore for $S = X$ we apply Theorem 4.3.5 to get $f = \mathcal{P}_\lambda F$ for some $F \in L^2(K/M)$.

If $\lambda = 0$, we apply Theorem 4.1.6 to get $f = \mathcal{P}_0 F$ for some $F \in L^\infty(K/M)$. □

**Theorem 6.3.2.** *For a measurable functions $f$ on $S$ if $\|\phi_{i\gamma_p\rho}^{-1}\Delta^k f\|_\infty \leq (4\rho^2/pp')^k$ for $k = 0, 1, 2, \ldots$, then $\Delta f = -(4\rho^2/pp')f$. If $S = X$ is a symmetric space, then $f = \mathcal{P}_{-i\gamma_p\rho}F$ for some $F \in L^{p'}(K/M)$.*

*Proof.* As in the previous theorem, one verifies that for any $\phi \in C^p(S)$, $|\langle f, \phi\rangle| \leq \gamma(\phi)\|\phi_{i\gamma_p\rho}^{-1} f(x)\|_\infty$. Therefore Theorem 5.2.1 can be used and we have, $\Delta f = -4\rho^2/pp' f$. Corollary 4.1.5 now shows for $S = X$ that $f = \mathcal{P}_{-i\gamma_p\rho}F$ for some $F \in L^{p'}(K/M)$. □

6.4. **Concluding Remarks.** Let us restrict our attention to the symmetric spaces where the characterization of the Poisson transforms is achieved.

(i) We fix $p \in (1,2)$. One notices that in Theorem B and its various analogues we conclude $f = \mathcal{P}_{-i\gamma_p\rho}F$ for some $F \in L^{p'}(K/M)$, while $\mathcal{P}_{i\gamma_p\rho}F$ for $F \in L^p(K/M)$ also satisfies the hypothesis (see (3.1.1) and thus is a candidate to be characterized. Indeed using the intertwining operator $I_p : L^p(K/M) \to L^{p'}(K/M)$ of Knapp and Stein ([27]), it was shown by Cowling Meda and Setti in [7] that for $F_1 \in L^p(K/M)$,

$$C_p\mathcal{P}_{i\gamma_p\rho}F_1 = \mathcal{P}_{-i\gamma_p\rho}I_p(F_1), \tag{6.4.1}$$

where $F = I_p(F_1) \in L^{p'}(K/M)$. Thus $\{\mathcal{P}_{-i\gamma_p\rho}F \mid F \in L^{p'}(K/M)\}$ is a bigger class which includes the Poisson transforms $\mathcal{P}_{i\gamma_p\rho}F_1$ for $F_1 \in L^p(K/M)$.

(ii) In the beginning of section 5 we have mentioned that the hypothesis of Theorem A an B (as well as their analogues) exclude the complex powers of the Poisson kernel. We have shown that the theorems of section 5 saves the situation. For the symmetric spaces we can also weaken the hypothesis in a different way to include those complex powers of Poisson kernel. This we shall describe below. We shall consider only the $L^2$-case, the $L^p$-case being similar. Let $\widehat{K}_M$ be the set of irreducible unitary representations of $K$ which contains an $M$-fixed vector. For $\delta \in \widehat{K}_M$, let $f_\delta$ be the $\delta$-isotypic component of a suitable function $f$ on $X$. Precisely, $f_\delta(x) = d_\delta \chi_\delta * f(x)$ where $\chi_\delta, d_\delta$ denote respectively the dimension and trace of $\delta$. We note that if $f(x) = e^{-(i\lambda+\rho)H(x^{-1}k)}$ for some $\lambda \in \mathbb{R}^\times$ then $f_\delta$ decays like $\phi_\lambda$ (see [23, 3.11]) and hence is in $L^{2,\infty}(X)$ (see section 2). In view of this we formulate the following:

**Theorem 6.4.1.** *Let $f$ be a measurable function on $X$ satisfying for some $\alpha > 0$, $\|\Delta^k f_\delta\|_{2,\infty} \leq C_\delta(\alpha^2+\rho^2)^k$ for all $k \in \mathbb{Z}$ and $\delta \in \widehat{K}_M$. Then $\Delta f = -(\alpha^2+\rho^2)f$.*

Notice that the decomposition of functions and distributions in $K$-types suggests an easier way to prove the main results for the symmetric spaces. We have noted this in Remark 5.1.2.

(iii) It should be possible to generalize some of the results considered in this paper to higher rank Riemannian symmetric spaces. We shall take it up in future.

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